# Reflections from Classroom Activities of Teachers Teaching at Different Grade Levels on the Development of Algebraic Thinking: Generalised Arithmetic

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**Abstract:** This study investigated the classroom teaching activities/practices of teachers working at different grade levels for the development of algebraic thinking in the context of generalised arithmetic. Case study method, one of the qualitative research designs, was used in the study because it provides the opportunity to examine the situations that take place in natural environments in detail and in depth. The study was conducted with a total of eight secondary mathematics and classroom teachers teaching at the 3rd, 4th, 5th and 6th grade levels. The lesson environments of many learning outcomes related to algebraic thinking and determined by expert opinion were observed at different time intervals throughout the 2021-2022 academic year. In order to prevent data loss regarding the observations, the observations were recorded with video and then the dialogues between the teacher and the students were transcribed. In addition, the observation data were supported by informal interviews with the teachers to clarify the situations that were not understood after the observed lessons. The data were analysed with the descriptive analysis technique, which is one of the qualitative data analysis techniques, taking into account the generalised arithmetic sub-theme in the theoretical framework of algebraic thinking themes created with the support of the literature and expert opinion, and then the results were supported by discussions based on the literature. As a result, it was determined that secondary mathematics and classroom teachers give very little place to teaching activities/practices to support algebraic thinking in the context of generalised arithmetic and in this sense, they cannot support the development of algebraic thinking very much. Therefore, it is very important to organise in-service training seminars for teachers on early algebra and algebraic thinking.

**Keywords:** Algebraic thinking, Genaralised arithmetic, Secondary mathematics and classroom teachers

## 1. Introduction

Learning algebra is seen by many students, especially in secondary education, as a challenge that leads to rejection of mathematics (Kaput, 1999; Kieran, 2004). Decades ago, Kieran (1989) warned that '*algebraic thinking is the area of greatest need in mathematical enquiry*' (p. 163). According to Brizuela and Blanton (2014), this challenge is largely due to the interpretation of Piaget's theory. According to Piaget (1969), students' cognitive development takes place in stages and the developmental stage for formal or abstract thinking begins around the age of 11 and is consolidated around the age of 15. This perspective suggests that primary school students are not yet ready to move from concrete procedural thinking to formal or abstract thinking, which has traditionally led to the postponement of algebra teaching in curricula until the early years of secondary school (Quevedo Gutiérrez & Llinares, 2021). As a matter of fact, the transition from concrete arithmetic thinking to more abstract algebraic thinking, which is required in secondary school and later grades, has become an obstacle for students' mathematics learning (Bekdemir & Işık, 2007; Carpenter, Levi, & Farnsworth, 2000; Knuth, Stephens, Blanton, & Gardiner, 2016). This problem has led educators and mathematics education researchers to consider a long-term and in-depth algebra reform (Kaput, 1999). In recent years, mathematics education researchers have recognised that algebra has a place in the early grades (Blanton, 2008; Blanton & Kaput, 2005a, 2005b; Carpenter, Franke, & Levi, 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006; Kaput, 1999; Kieran, 2004; Schifter, Russell, & Bastable, 2009). Kaput (1999) defines the path to this reform as 'the introduction of algebra into the mathematics curriculum from the very beginning of school' (p. 134). However, this process is a long and arduous task. This is because it is not easy to encourage students to think algebraically and to maintain and extend this way of thinking in the following years during this process, which involves a longitudinal learning trajectory. Indeed, researchers and curriculum designers have agreed that students should engage with algebra continuously and productively throughout their education (National Council of Teachers of Mathematics [NCTM], 2020). This decision is considered correct for at least two reasons: (1) the exclusive focus on arithmetic (numbers) and calculations in elementary mathematics limits the conceptual development of mathematical ideas in the early grades (Blanton & Kaput, 2005a); (2) students' abrupt introduction to algebra in middle school through traditional courses results in serious difficulties in understanding algebraic concepts (Cai & Knuth, 2005). Many studies have emphasised that algebraic thinking of primary school students can be supported while they are engaged in arithmetic calculations (Howe, 2005; Inprasitha, 2016; Lins & Kaput, 2004). For this reason, both algebra and algebraic thinking have been considered as a central topic of

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mathematics education by many researchers, educators and curriculum designers in recent years. In this context, studies on early algebra and algebraic thinking will make a significant contribution to the literature in this field.

### 1.1. Algebraic Thinking and Generalised Arithmetic

Although it is related to algebra, algebraic thinking has a broader and different meaning than algebra. This is because algebraic thinking is a special form of mathematical thinking and it is not limited to the learning domain of algebra. Driscoll (1999) drew attention to the difficulty of giving a definition of algebraic thinking because algebra is related to many mathematical concepts. Algebraic thinking not only opens the door to the abstract thinking required for algebra, but also serves as a protective role for individuals' efforts towards their progress in mathematics and different disciplines (Greenes, Cavanagh, Dacey, Findell, & Small, 2001), and includes mental activities related to thinking about the problems they encounter in their daily lives, making predictions/assumptions and producing solutions. There are different opinions about what algebraic thinking is and how it develops/emerges. Algebraic thinking in an arithmetic environment involves viewing arithmetic with 'algebraic eyes' (Subramaniam & Banerjee, 2011) and this is called algebraisation. Cai and Knuth (2011) define this as the nature of thinking required for learning algebra related to conceptual domains in elementary and middle school mathematics. It is not about introducing algebra early, but about helping students learn to reason algebraically and begin to acquire a symbolic (algebraic) language to express and justify their ideas (Blanton, 2008). Indeed, just as many civilisations solved algebra problems before the existence of algebraic notation, students can work with variables and arithmetic rules before algebra is taught (Harper, 1987; p. 670). From this perspective, Carraher and Schliemann (2007) define algebraic thinking as the psychological processes involved in solving problems that mathematicians can easily express using algebraic notation. In this sense, Carraher and Schliemann's definition emphasises the implicit cognitive processes that need to be taken into account when young learners engage in problem solving (such as recognising structural relationships and making generalisations) and suggests that some of these processes may involve variables and rules of arithmetic. Carraher and Schliemann (2007) also claimed that algebraic thinking involves generalising relationships between numbers and is different from arithmetic thinking. Kieran (2004) defined algebraic thinking as thinking that can be performed without using letter-symbolic algebra in any of the activities such as analysing the relationships between quantities, noticing structure, examining change, generalising, problem solving, modelling, verifying, proving and predicting, that is, developing students' thinking styles with the help of activities that are not specific to algebra.

As can be seen from the definitions, it is important to answer the question '*What kind of algebraic concepts can children learn in teaching environments that support algebraic thinking (early algebra)?*' (Kaput, Blanton, & Moreno, 2008, p. 18) in order to prepare them for formal algebra in later grades. The distinction between algebra in the early grades and traditional school algebra (algebra comes after arithmetic) has raised the question of what algebra is and what kinds of thinking should be considered algebraic (Bell, 1996) and the need to reconceptualise these concepts (Kaput, 1998). To this end, several recommendations have been put forward that emphasise important aspects of algebra (Bednarz, Kieran, & Lee, 1996; Kaput, 2008; Kieran, 1996; Usiskin, 1988). Among these recommendations, Kaput, a pioneer of the early algebra approach, provided a useful framework to guide this study. In his study, Kaput (2008) stated that algebraic thinking consists of two main themes: (1) making generalisations and expressing these generalisations in formal or known symbol systems, (2) reasoning about these symbolic forms. Kaput (2008) explained that the second theme is usually developed after the first theme is developed. In other words, by using symbols as a tool in the algebraic reasoning process, a relational understanding should be developed first, and then the ability to perform operations with symbols should be taken into consideration. Kaput (2008) stated that these two basic algebraic thinking themes overlap with the following three topics within the K-12 curriculum and thus evaluated different aspects of algebra: Generalised arithmetic, Functional thinking, Modelling languages.

Generalised arithmetic has different meanings from arithmetic involving numbers and numerical calculations, even though arithmetic is mentioned in its name. In this sense, generalised arithmetic means '*helping children to see, identify and justify patterns and regularities in operations and properties of numbers in order to move beyond arithmetic to algebraic thinking*' (p. 12) (Blanton, 2008). While Carpenter et al. (2003) defined generalised arithmetic as performing operations with numbers and reasoning about the properties and relationships of numbers, Kaput (2008) emphasised that generalised arithmetic involves performing calculations with numbers, examining these calculations, recognising their relationships, making generalisations from these relationships, reasoning about these generalisations and using them in problem solving. As can be understood from Kaput's (2008) definition, generalised arithmetic involves looking at the structure of the operations in addition to focusing on the results of the operations. According to Chimoni, Pitta-Pantazi and Christou (2018), generalised arithmetic refers to the definition of relationships between numbers, manipulation of operations and their properties, and transformation and solution of equations.

As can be understood from these definitions, generalised arithmetic is about going beyond calculations with specific numbers; defining patterns in arithmetic and thinking about the mathematical structures underlying arithmetic (generalising the properties related to numbers and operations involving numbers) (Akkan, 2009). For example, it is easier for students who examine the relationships in numerical operations such as 28 = 4 × 7 = 4 × (2 + 5) = (4 × 2) + (4 × 5) = 8 + 20 = 28 to make sense of algebraic expressions such as '5a + 8a = a × (5 + 8) = 13a' and to reach generalisations about the distributive property such as 'a × (b + c) = (a × b) + (a × c)'. In this sense, the variables used by the students who perform the transformation and generalisation actions in this transformation process help them make the transition from algorithmic calculation to generalisation. Because it is obvious that such arithmetical studies, which students are familiar with especially in primary school classes, will form the basis for algebraic thinking. Studies on generalised arithmetic are mainly concerned with a) the basic properties of numbers and operations, b) the relationships in a class of numbers and the results of calculations, and b) the relationships between operations (Öztürk, 2021). Indeed, algebraic thinking can be developed by generalised arithmetic in different ways (Kaput, 2008; Ontario Ministry of Education [OME], 2013; Van de Walle, Karp & Bay Williams, 2013): Generalisations of basic properties of arithmetic, Generalisations of arithmetic other than basic properties, Inverse operations. These types of generalisation activities involve generalising arithmetic relations involving number and operation properties (both relations in basic properties such as change, combination, dispersion and arithmetic relations other than basic properties) and relations between operations that are inverse of each other. This is because algebraic thinking is related to recognising and analysing regularities or relations in arithmetic operations, generalising these regularities or relations and performing operations with unknown quantities (Kaput, 2008).

### 1.2. Significance and Purpose of the Study

Although it is thought that algebraic thinking cannot be developed in the early grades, there are many studies in the international literature that algebraic thinking can be developed in the primary grades (Blanton, Levi, Crites, Dougherty, & Zbiek, 2011; Blanton, et al., 2015; Kieran, 2007; Schifter & Bastable, 2008). Blanton, et al. (2011) stated that elementary students develop important habits of mind when they consistently have some experience with algebraic reasoning and that these students gain a much deeper mathematical understanding compared to those who have experiences that focus on arithmetic competence and thus are better prepared for formal algebra learning. NCTM (2000), on the other hand, stated that high school education is insufficient for the development of algebraic thinking in students, and that this insufficiency stems from the teaching of arithmetic and algebra in elementary and middle school. In line with this recommendation of the researchers and NCTM, it is important to analyse in detail the teaching of teachers in our country for the development of algebraic thinking in order to teach the basic topics related to formal algebra effectively.

The view that algebra is a core subject throughout elementary and middle school mathematics is increasingly accepted and welcomed. However, studies have shown that major advances have not yet been made in teaching materials, classroom activities, and teachers' mindsets (Carraher & Schliemann, 2018). However, for algebraic thinking to become an important goal in early mathematics teaching, teachers need to change their beliefs and acquire new ways of engaging their students in activities. This is difficult to achieve without fundamental improvements in the training of future mathematics teachers. In this sense, this study is important in terms of identifying the situation of teachers and providing direction for teacher education. On the other hand, in both national and international contexts, the question 'How can teachers or prospective teachers be given a good start in developing the basic algebraic knowledge necessary for teaching?' (Fey et al., 2007, p. 27) remains unanswered. A widely accepted key recommendation for answering this question is to reconceptualise algebra teaching and learning from a K-12 perspective so that students have a long-term and sustained algebra experience starting in the elementary grades (Kilpatrick, Swafford, & Findell, 2001; NCTM, 1989, 2000). In theory, such an approach would allow children's algebraic thinking to develop more naturally by utilising their natural intuitions about structure and relationships (Mason, 2008) from the beginning of formal education. At the same time, this development of algebraic thinking will improve children's success with more formal mathematics, particularly algebra, as they progress to the middle grades and beyond. Kaput (2008) recommended a good start, a more in-depth curriculum restructuring in the context of algebra, changes in classroom practice and assessment, and new arrangements for teacher training. However, early algebra is not included in the mathematics curriculum of our country as a learning area for primary school grades (Milli Eğitim Bakanlığı [MEB], 2018), and studies on early algebra, including changes in classroom practice and assessment and adjustments in teacher training, have only recently been conducted in our country. Therefore, the findings of this study will be very important.

The source of difficulties in algebra is attributed to students' lack of experience in arithmetic. In classrooms where early algebraic thinking is supported, students can recognise mathematical relationships, make predictions, generate and verify mathematical ideas, do proof work in an age-appropriate manner, and generalise these ideas (Blanton & Kaput, 2005a; Schifter & Bastable, 2008). Indeed, students' understanding of mathematical concepts, problem solving skills, dispositions towards mathematics and beliefs are shaped by the teachers they encounter in their school life (NCTM, 2000). Teachers have a great role in supporting these activities related to early algebraic thinking. According to Hunter, Anthony and Burghes (2018), the design and implementation of instructional approaches as well as specific pedagogical measures are crucial in developing early algebraic thinking. When implementing the current curriculum, teachers need to identify materials and assign tasks that will enable the development of early algebraic thinking (Blanton & Kaput, 2005a). However, teachers have little experience in teaching approaches and integrating algebraic thinking in early grades and have difficulties (Blanton & Kaput, 2005a). In fact, Hunter, Anthony and Burghes (2018) categorised these difficulties under two headings in their study: (1) lack of general understanding, lack of appropriate mathematical language, and inability to use age-appropriate symbols, and (2) teachers' inexperience in activities to develop early algebraic thinking. For these reasons, teachers are needed to develop early algebraic thinking, and teachers' knowledge in this area is considered to be the key to effective teaching both in early algebra and in a broader context (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997; Blanton & Kaput 2008; Shulman 1987). In this context, it is important to determine the classroom teaching activities of both secondary mathematics and classroom teachers in the context of early algebraic thinking. In this study, it was aimed to examine in-depth the teaching activities of secondary mathematics and classroom teachers for the development of algebraic thinking in the context of generalised arithmetic. In line with this purpose, the following problem and sub-problems parallel to this problem will be sought in the study:

*What kind of in-class teaching activities do secondary mathematics and classroom teachers support their students for the development of algebraic thinking in the context of generalised arithmetic?*

## 2. Method

### 2.1. Research Model

In this study, which aims to descriptively reveal the in-class teaching activities of secondary mathematics and classroom teachers regarding the development of (early) algebraic thinking in the context of generalised arithmetic, the case study method, one of the qualitative research designs, was used. Case studies are studies that describe the events that take place in their natural environment in detail and in depth with different data collection tools under time and space constraints (Hancock & Algozzine, 2006). According to Creswell (2012), a case study is a study in which situations are examined in depth over a certain period of time through observations, interviews, reports, documents and audio-visual materials.

### 2.2. Design and Conduct of the Study

This study was conducted in three stages: preparation, implementation and evaluation. In the preparation phase, a literature review on early algebra and algebraic thinking was conducted to determine the research problems and the appropriate research design. The process of forming the four main themes and their contents in the conceptual framework of the research was supported by many studies reviewed, and the theme of generalised arithmetic was focused. In this process, the learning outcomes related to mathematics in the 3rd-6th grades were examined and the learning outcomes that were thought to affect the development of early algebraic thinking of students at different levels of education in the context of generalised arithmetic were determined by taking expert opinion. The implementation phase of the study was carried out with a total of eight secondary mathematics and classroom teachers. Teachers were informed about both the purpose and the process of the study, and they were asked to inform the researcher about the lesson day and time of the acquisitions to be observed within the plan. In this process, the start and end dates of the observations made with the teachers, the duration of the observations and the number of objectives observed are given in Table 1. In addition, in order to avoid problems during the implementation process, continuous communication was maintained with the teachers, frequent reminders were made, and the lessons containing the acquisitions were carefully followed. The lessons were video recorded to prevent data loss during the observations. In order to support the data obtained during the observations, noteworthy situations were noted. İnformal interviews were conducted with the teachers to clarify the situations that were not understood as a result of the data obtained with the videos and the findings were made ready for evaluation. In addition, the lessons were videotaped to prevent data loss during the observation. In case of overlapping lessons to be observed, the researcher asked the teacher to record the lessons, the video was carefully watched by the researchers on the same or the next day, and informal interviews were conducted with the teachers for any unclear situations. These data were classified using the descriptive analysis technique, which is a qualitative data analysis technique, taking into account the generalised arithmetic sub-theme, which is one of the themes of early algebraic thinking, and the findings were detailed by adding quotations and visuals. Then, the results of the study were obtained by discussing the data obtained with the support of the literature.

### 2.3. Research Group

The research group consisted of secondary mathematics and classroom teachers working in a district in Konya province. The 3rd-6th grade students attending the mathematics lessons of the teachers were also implicitly included in the research group. The four teachers determined for the research group were selected using criterion sampling method, which is one of the purposeful sampling methods. As a matter of fact, it is a characteristic of the purposive sampling method to select the people and places that will best help the researcher in understanding the phenomenon (Creswell, 2012). However, in criterion sampling, the sample is formed from people, events, objects or situations with the characteristics determined in parallel with the purpose of the research (Büyüköztürk, Kılıç Çakmak, Akgün, Karadeniz, & Demirel, 2017). According to Yıldırım and Şimşek (2018), in this sampling type, the researcher himself decides on the situations and participants to be studied and determines the criteria himself. In order to ensure diversity in the sample, teachers with different years of experience, gender, and both bachelor's and non-thesis master's degree graduates were tried to be selected. In addition, diversity was also ensured in terms of teachers working in different primary and secondary schools. In the study, two classroom teachers for each grade teaching in grades 3–4 and two secondary mathematics teachers for each grade teaching in grades 5–6 were studied and the demographic information of the participant teachers is presented in Table 1. The participants were given codes as $T3_1$, $T3_2$, $T4_1$, $T4_2$, $T5_1$, $T5_2$, $T6_1$ and $T6_2$, and the identities of the participants were kept confidential due to the ethics of the research.

**Table 1.** Information on Teachers Observed in Lesson Observations

| Teacher Codes | Gender/Year of Experience/Status of Education | | | Observation Start Date | Observation End Date | Observation Time (min) | | Number of Observed Learning Outcomes (LO) and Codes |
|---|---|---|---|---|---|---|---|---|
| | | | | | | Whole observation time | Observation time reviewed | |
| $T3_1$ | Male | 24 | Undergraduate | 16.09.2021 | 28.03.2022 | 1050 | 325 | 30 ($LO3_1$ – …– $LO3_{30}$) |
| $T3_2$ | Female | 22 | Undergraduate | 16.09.2021 | 13.01.2022 | 1080 | 280 | 30 ($LO3_1$ – …– $LO3_{30}$) |
| $T4_1$ | Male | 16 | Undergraduate | 13.09.2021 | 23.03.2022 | 945 | 785 | 27 ($LO4_1$ – …– $LO4_{27}$) |
| $T4_2$ | Male | 27 | Master's Degree without Thesis | 13.09.2021 | 03.01.2022 | 640 | 255 | 27 ($LO4_1$ – …– $LO4_{27}$) |
| $T5_1$ | Female | 10 | Undergraduate | 16.09.2021 | 03.11.2021 | 540 | 450 | 15 ($LO5_1$ – …– $LO5_{15}$) |
| $T5_2$ | Female | 14 | Undergraduate | 21.09.2021 | 26.11.2021 | 600 | 580 | 15 ($LO5_1$ – …– $LO5_{15}$) |
| $T6_1$ | Male | 7 | Master's Degree with Thesis | 16.09.2021 | 08.01.2022 | 450 | 275 | 15 ($LO6_1$ – …– $LO6_{15}$) |
| $T6_2$ | Male | 16 | Undergraduate | 23.09.2021 | 15.12.2021 | 400 | 325 | 15 ($LO6_1$ – …– $LO6_{15}$) |

* *$LO3_1$ – …– $LO3_{30}$: codes of the outcomes observed in the 3rd grade; $LO4_1$ – …– $LO4_{27}$: codes of the outcomes observed in the 4th grade; $LO5_1$ – …– $LO5_{15}$: codes of the outcomes observed in the 5th grade; $LO6_1$ – …– $LO6_{15}$: codes of the outcomes observed in the 6th grade*

### 2.4. Data Collection Tools

In this study, which aims to examine the teaching activities of secondary mathematics and classroom teachers regarding the development of algebraic thinking in the context of generalized arithmetic, unstructured observations were used to collect data. If a comprehensive, detailed and time-extended view of the participants' behaviors is desired in any setting, the observation method can be used (Bengtsson, 2016; Fossey, Harvey, McDermott, & Davidson, 2002). Therefore, observations are frequently used especially in studies examining knowledge and skills related to teaching. The non-participant observer is an "outsider" who sits on the sidelines or in a place that provides him/her with an advantage for observation and watches and takes notes of the phenomenon being studied. This role requires less access than the participant role, and the responsible persons and participants in the research setting may feel more comfortable in this situation (Creswell, 2012). In this context, the researcher assumed the role of a non-participant observer (observation role as an observer) and observed the teaching environments of middle school mathematics teachers in which the outcomes thought to be related to early algebraic thinking were addressed during the fall and spring semesters of the 2021-2022 academic year, and recorded these observations with video to prevent data loss. In addition, in cases where the researcher had questions about the findings obtained as a result of the observation and in order to support the observations, he conducted conversational (informal) interviews with the teacher he observed after each observation. The aim of these informal interviews is to make the current situation more understandable. According to Yıldırım and Şimşek (2008), this interview is generally used in studies where the researcher directly participates in the environment for the purpose of observation, the questions are asked in the natural flow of the interaction, and the interviewee may not even realize that he/she is being interviewed. In addition, the researcher took field notes in the classroom environment regarding the teachers' in-class teaching activities, and the content of these field notes included details regarding the teacher's explanations of the concepts and the researcher's comments regarding the observation.

### 2.5. Data Analysis

In order to analyse the data of the research, a part of the theoretical framework in Figure 1 (Memişoğlu Çoban, 2023), which was created by examining many studies in the literature, was used. The theoretical framework presented in Figure 1 consists of the themes of generalized arithmetic, functional thinking, modeling languages, and meaningful use of symbols. However, in the article, data were analyzed only according to the sub-theme of generalized arithmetic of this theoretical framework. Indeed, algebraic thinking can be developed by generalised arithmetic in different ways:

*Generalizations about basic properties in arithmetic:* Such generalizations involve generalizing arithmetic relations involving properties of number and operations (relations that result from observing how operations behave and are related to each other) and reasoning about these generalizations (about the structure of arithmetic expressions rather than the results of calculations). In particular, this includes understanding how to generalize basic properties (e.g., commutation, union, dissipation, etc.) and how these basic properties can be used in computations and transformations of algebraic expressions.

*Arithmetic generalizations other than fundamental properties:* Such generalizations are related to relations in number classes and results of calculations. For example, they stated that the properties of relations in number classes and the properties of numbers that express the results of calculations such as odd/even numbers are concepts related to generalized arithmetic.

*Operations that are the inverse of each other:* Since inverse operations have an important role in the mathematical modeling process or in the abstraction of algebraic elements such as equality, equivalence, equation, inequality, which are products of the modeling process, many curricula introduce the learning domain of numbers (arithmetic) (subtraction is the inverse operation of addition and vice versa, division is the inverse operation of multiplication and vice versa) from the first grade of primary school.

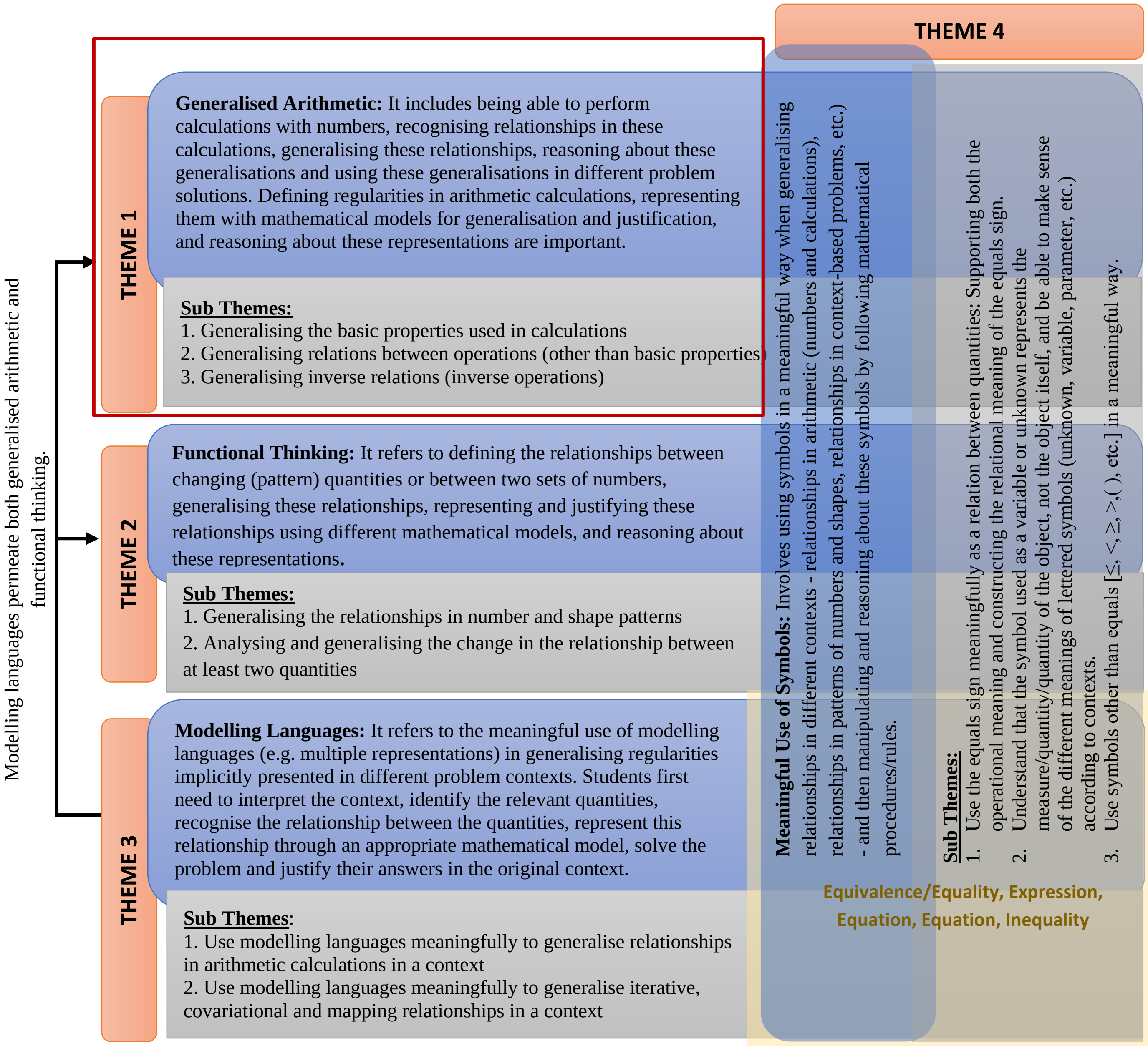



**Figure 1.** Theoretical Framework Used in the Study (Memişoğlu Çoban, 2023)

In this sense, since observations were used in the investigation of the classroom teaching activities of secondary mathematics and classroom teachers regarding algebraic thinking in the context of generalized arithmetic and the obtained data were analyzed according to the above theoretical framework prepared with the support of the literature, the descriptive analysis technique was used in this study. Descriptive analysis is a type of qualitative data analysis that involves the processing, classification, summarization, and interpretation of qualitative data obtained through various data collection techniques (interview, observation, etc.) in accordance with a previously determined theoretical framework or themes (Yıldırım & Şimşek, 2018). In this type of analysis, the researcher can frequently include direct quotes in order to reflect the views of the individuals he/she has interviewed or observed in a striking way and can make comparisons between the facts when necessary by establishing a cause-effect relationship between the findings.

In qualitative research, methods such as long-term interaction, expert review, and participant confirmation are used to ensure credibility (Yıldırım & Şimşek, 2018). Yıldırım and Şimşek (2018) stated that interacting with data sources for a long time in research increases validity. For this reason, keeping the observation periods long in the research positively affected the validity. In addition, a mathematics educator who is an expert in algebra education approached the study critically and made suggestions, and this situation increased the quality of the research. In qualitative research, concentrated description and purposeful sampling are used to ensure transferability (Yıldırım & Şimşek, 2018). According to Creswell (2012), concentrated description can be achieved by providing detailed and sufficient information when describing a situation or theme, and the data should not be manipulated and direct quotations should be included. In this study, in order to ensure transferability, all data obtained in the context of early algebraic thinking and generalized arithmetic were evaluated and the study group was selected according to the purposive sampling technique. Confirmability in qualitative research is related to confirming the results obtained as a result of the process steps of the research with raw data, that is, re-examining the raw data stored by the researcher and the codes formed as a result of data analysis when necessary (Yıldırım & Şimşek, 2018). In this study, in order to ensure confirmability, all steps of the research were tried to be presented in detail, and each stage of the raw data and sub-themes reached as a result of the analysis were transferred to the computer environment and stored. In qualitative research, a triangulation strategy (working group, data collection, triangulation analysts, etc.) can be used to ensure internal validity (Patton, 2002; Yıldırım & Şimşek, 2018). According to Patton (2002), the triangulation "analyst strategy" involves two or three people analyzing the obtained data separately and comparing their results. In this study, in order to ensure internal validity, the data were analyzed separately by the thesis writer, the advisor, and an expert mathematics teacher, and the findings were compared. In qualitative studies, reliability or consistency can be achieved through the compatibility of the data with each other and descriptions that convince the reader (Merriam, 2018). According to Creswell (2012), reliability is capturing the stability between the results of more than one researcher in a study. According to Miles and Huberman (1994), reliability between researchers is calculated with the formula,

$$\text{Reliability coefficient} = \frac{(\text{Consensus})}{(\text{Disagreement} + \text{Consensus})} \cdot 100$$

and the agreement between researchers should not be less than 80%. The reliability coefficient in the study was calculated as 91%, and the researchers came together again for disagreements, discussed them, and reached an agreement.

### 2.6. Research Ethics

All participants provided informed consent prior to their involvement in the study. They all were informed about the aim of the study and their right to withdraw at any time without consequence. Since the data were collected before 2020, no additional ethical approvement was required.

## 3. Findings and Discussion

The findings obtained as a result of the observation of teachers teaching at the 3rd-6th grade levels were discussed in the context of the generalised arithmetic component of algebraic thinking and the data obtained are presented and discussed below. In this context, the findings obtained from the observed teaching environments of the teachers are presented according to three sub-themes of generalised arithmetic - findings related to generalisations of basic properties in arithmetic, findings related to generalisations of arithmetic other than basic properties, findings related to relations between inverse operations.

### 3.1. Activities Performed by Teachers in the Context of Generalising Basic Properties in Arithmetic

The data obtained from teachers teaching at different grade levels were obtained as a result of observing the classroom environments in which the learning outcomes coded $LO3_{10}$, $LO3_{11}$, $LO4_{7}$, $LO4_{16}$, $LO5_{10}$, $LO6_{3}$ were addressed. Most of these learning outcomes involve the generalisation of arithmetic relations including the basic properties of numbers and operations, i.e. the relations obtained as a result of observing how operations behave and how they are related to each other, and reasoning on these generalisations. In this context, both $T3_1$ and $T3_2$ classroom teachers did not engage in any activity for their students to discover and generalise basic properties such as change, merger, etc. by making use of the relationships between numbers and operations in any outcome, including outcome $LO3_{11}$. For example, as can be understood from the dialogue below, although it was observed that $T3_1$ did not give the information directly while talking about the property of change, the teacher stated the generalization himself by using a single example and only the relationship between two numbers, but he did not design a teaching activity in which his students could reach the generalization by using two or more examples. In other words, the teacher did not take any action involving the students to express the relationship between the two natural numbers in their own sentences and expressed the generalisation directly.

> $T3_1$: *Firstly, we add the ones and write them in the ones digit, add the tens and write them in the tens digit, add the hundreds and write them in the hundreds digit. This can be addition with or without elimination. The sum does not change when the place of addition changes. For example, let's take the addition operation 124 + 263. We call 124 the first addition and 263 the second addition. Now let's add 4 and 3 more?* [Students: 7]
> $T3_1$: *2, 6 more?* [Students: 8]
> $T3_1$: *1, 2 more?* [Students: 3]
> $T3_1$: *How many is that?* [Students: 387.]
> $T3_1$: *I was bored, so I did this... Let's change the places of the first addition and the second addition and add them, let's see if the result will change. What is the total, students?* [Students: 387.]
> $T3_1$: *Did the result change? No... Then the result does not change when the place of the additions is changed. Did we understand?*

Similarly, in none of the objectives, including the $LO3_{11}$ coded objective of the $T3_2$ classroom teacher, she did not carry out any activity for her students to generalise the basic properties such as change, merger and dispersion by making use of the relationships between numbers and operations. In this context, it was understood that classroom teachers $T3_1$ and $T3_2$, who did not carry out any activity to enable students to make assumptions or predictions about the property of change, did not sufficiently support the component related to generalising the basic properties of numbers and operations.

Classroom teacher $T4_1$ did not carry out any activity or practice that would enable her students to reach a generalisation about the change property of addition, she presented the rule to the students in a memorised way with a statement such as "*In addition, the sum does not change even if the added numbers change places*" and then tried to verify the validity of the rule through the operation 3708 + 5693. As can be understood from the dialogue, this teacher could not create a suitable classroom environment in the context of algebraic thinking for students to find a relationship and express the generalisation in their own sentences through different examples about the change property of addition.

> $T4_1$: *In addition, the sum does not change even if the added numbers change places. For example; let's take 5693 + 3708. Let's do this operation together... Let's write the names: 5693 is the first addition and 3708 is the second addition. The result is 9401... Now let's change their places: 3708 + 5693. Now let's add them... Has the result changed?* [Students: Unchanged.]
> $T4_1$: *We can also do this in a 5-additive operation. In that case, we can change the place of additions as we want, but the result, that is, the sum, never changes.*

This practice of the teacher is a traditional teaching style, and this classroom teacher displayed a similar approach while teaching the lesson related to outcome $LO4_{16}$, which is about the changing property of multiplication. From the dialogue below, it is understood that while teaching multiplication, teacher $T4_1$ had the students write a title such as changing the order of multipliers and gave rule-oriented information about the changing property of multiplication and common bracketing. As a matter of fact, classroom teacher $T4_1$ started the lesson with a rule statement such as "*In multiplication, even if the multipliers change places, the product, that is, the result, never changes...*" and then showed the correctness of this rule with two different examples. Then, by referring to addition, he drew attention to the fact that there is a similar rule in addition.

> $T4_1$: *In multiplication, even if the factors change places, the product, i.e. the result, never changes. I will show this rule with multiplication operations. For example, let's take 9 × 7. 9 is the first factor and 7 is the second factor. 9 × 7 = 63. Well, let's change their places to 7 × 9. 7 is the first factor and 9 is the second factor. 7 × 9 = 63. Then the result will never, ever change. Well, it can also be like this. In the same way, when the addends were swapped in addition, did the sum change?* [Students: No…]
> $T4_1$: *Here too, even if the factors are swapped, the result of the multiplication operation never changes... Let's write it down...*

On the other hand, the following dialog reveals $T4_1$'s justification of the rule about the use of parentheses and why this rule is necessary. In the rest of the lesson, it was observed that teacher $T4_1$ included different examples to show the correctness of the rule and continuously warned that the operation inside the parenthesis should be done first. However, it was also determined that $T4_1$ classroom teacher did not emphasize the combining property of multiplication reflected in the expressions $(15 \times 8) \times 26 = 3120$ and $15 \times (8 \times 26) = 3120$.

> $T4_1$: *In the multiplication of three or more numbers, the two determined or desired numbers are enclosed in parentheses. In such cases, the operations in parentheses are done first. This both allows us to find the result more easily and makes multiplication easier. We were writing and multiplying two*

*numbers side by side, but when they are side by side, you can write and multiply as many numbers side by side as you want, but when two numbers come side by side when they come side by side, it is necessary to use parentheses to make it easier. 15 × 8 × 26 yes I have 3 operations, right? To make such an operation easier, I put the two numbers I want in parentheses. (15 × 8) × 26 = 120 × 26 = 3120 and 15 × (8 × 26) = 15 × 208 = 3120. See? Let's do one more. For example, I can do it like this: 8 × (15 × 26) = 8 × 390 = 3120...Now it is difficult to multiply three numbers without parentheses, but if you put two of the numbers in parentheses like this, the process becomes really easy. First we will do the operation inside the parenthesis and then we will do the operation outside the parenthesis. Let's multiply them all separately and see if we get the same result... Look, I always get the same result.* [Student: What would we do if there were 5 factors?]

$T4_1$: *Let me explain with a simple example: 2 × 7 × 9 × 12 × 3 = (2 × 7) × ( 9× 12) × 3 = 2 × (7 × 9) × (12 × 3)…* [Student: Well, if there are 4, there will be nothing left out.]

$T4_1$: *It doesn't necessarily have to be outside... For example, 8 × 7 × 5 × 9 = (8 × 7) × (5 × 9) = 56 × 45. For example, I can do (7 × 5) × (8 × 9). Let's take 35 × 72. (7 × 9) × (8 × 5) = 63 × 40. If there are seven numbers, one number is left out. If there are 4 numbers, there will not be. But there is this, you will definitely, definitely do the inside parenthesis first and then continue the process.*

In this sense, just like the third grade teachers, teacher $T4_1$ did not carry out any activity or practice for students to generalize the relationships in numbers and calculations. Similarly, it is understood from the dialog below that teacher $T4_2$ did not allow the students to discover the changing property of multiplication while teaching the lesson related to learning outcome $LO4_{16}$ and presented the information directly to the students with a sentence such as "*Even if the multiplier changes places in multiplication, the product, that is, the result, never changes*." It is also seen that the teacher solved examples to reinforce the changing property of multiplication by giving the product of different numbers.

$T4_2$: *So what happens if we change the first addition and the second addition in addition? Let's swap these two. 1071 goes up and 2482 goes down* [The teacher writes the two numbers one below the other and draws an addition line]...*So we took the second addition up and made it the first addition. We made the first summation the second summation...*[The teacher adds the two numbers one under the other with the same method and finds the same result]... *As you can see, the result here and the result here are the same. Let's make an information cloud and I write inside the information cloud. "In addition, changing the places of the added numbers does not change the result." How did we know? We switched and added them and got the same result. ... Shall we add another knowledge cloud there?* [Students: Yes...]

$T4_2$: *We say that the number of terms collected in addition can be as many as we want...*

In the continuation of the lesson, $T4_2$ had the students write a title such as parenthesized operation and operation priority and presented the topic directly to the students without designing any discussion environment. We also observe the rote memorization and traditional teaching style of $T4_2$ classroom teacher, who does not allow her students to express the generalization in their own language, while explaining the change property of addition. For example, teacher $T4_2$ tried to teach the change property of addition through the example of "1071 + 2482", but without questioning whether the students understood this basic property conceptually, he wrote a rule as "*The change of the places of the numbers added in addition does not change the result*". Similarly, it was observed that teacher $T4_2$ directly taught that addition can be done with more than two numbers with a rule such as "*We say that the number of terms added in addition can be as many as we want*." and designed a lesson without allowing students to question and think about the relationships. Although there is no learning outcome related to the change property of subtraction in our country's 2018 1st-4th grade mathematics curriculum, the $T4_2$ teacher also gave direct information to the students about the lack of change property in subtraction. As a matter of fact, it is understood from the following dialog that the students directly answered "no" to the teacher's question "*Well, can we substitute the subtraction for the subtraction?*" without thinking, and neither the students questioned the teacher nor the teacher questioned the students.

$T4_2$: *Now, what do we call the number over there? The number here is the subtraction and the number here is the difference. Now, the issue we will pay attention to in this subtraction process is that the number subtracted from the subtracted number must be smaller. So, can we interchange the subtracted and the subtracted in subtraction?* [Without giving students the floor...] [Students: No]

$T4_2$: *No, we cannot change it because the subtraction cannot be made from the subtraction since the number is small. So there is no substitution feature. We could also add, we could substitute the additions and the result did not change. But not in this case. Can you subtract two thousand from a thousand? No.*

In the informal interview about why the students answered no to the question of whether subtraction provides the property of change, it was understood that the students gave this answer depending on the reaction in the class and that they did not have any information about the property of change of subtraction before. This situation revealed that students unconditionally accepted the information presented to them without questioning or reasoning. In this sense, it was observed that $T4_2$, just like $T4_1$, adopted a rote memorization approach in the teaching of all related outcomes and did not engage in any activities or practices to support algebraic thinking.

Although there was no acquisition about the basic properties of numbers and operations in the 5th grade curriculum, mathematics teacher $T5_1$ emphasized the ineffective and absorbing element property in multiplication in a part of the lesson as in the dialogue below.

> $T5_1$: *Now everyone look here. I am drawing a friend here, this is his belly and his shirt buttons are stretched. He likes to eat numbers the most...* [Here, the teacher explains the absorption property in multiplication with this kind of story.]... *987×0=0... There is a man who has no appetite and does not eat anything...* [Here, the teacher explains the ineffective element property in multiplication.] *305×1=305...*

As can be understood from the dialogue, it is seen that $T5_1$ mathematics teacher tried to explain his lesson with the storytelling technique and depicted the numbers 0 and 1 as a person's navel. After the depiction, $T5_1$ mathematics teacher tried to make his students understand the property of the absorbing and ineffective element in multiplication by giving a few simple examples such as "$987 \times 0 = 0$" and "$305 \times 1 = 305$". However, it is noteworthy that the teacher gave the information directly to the students, and there was no in-class teaching activity that would enable the students to discover that the numbers 0 and 1 are respectively the absorbing and inefficient elements in multiplication. In addition, it was determined that teacher $T5_2$ did not engage in any in-class activities or practices in the context of determining and generalizing the relationships between the basic properties in arithmetic during the observation period.

It was observed that mathematics teacher $T6_1$ presented the property of dispersion to students with animal depictions in a lesson environment in which he addressed outcome $LO6_3$. It was observed that this mathematics teacher started the lesson by making sentences that would arouse curiosity in the students before introducing the topic. In this process, students communicated with the teacher with a question such as "How does it work, teacher?" and a discussion environment started. Afterwards, the teacher tried to make the subject more concrete by replacing the numbers with bees and flowers. The mathematics teacher $T6_1$ tried to verify the validity of the property by giving different examples containing the property of dispersion to the students. During the rest of the lesson, the mathematics teacher $T6_1$ gave students the opportunity to ask questions and gave them immediate feedback in the form of confirmation or falsification. The dialog of this teacher is presented below:

> $T6_1$: *Normally, it is a "( )" sign. There is a parenthesis and outside of it there is a number waiting to be multiplied. What do we normally do? We normally do the inside of the parenthesis and then multiply the number outside, for example, in the number $3 \times (8 - 4)$, you first do the parenthesis, then you do 4, then you multiply it by 3, $3 \times 4$, what is the answer? 12 Is there a solution to this problem in another way?* [The teacher mentions the bee-flower story below]... *Here is the logic of this distributive property, young people, if there are two numbers containing addition or subtraction inside the parenthesis, you can do the multiplication outside by distributing and the result is the same, incredibly. For example, $3 \times (8 - 4) = 3 \times 8 - 3 \times 4 = 24 - 12 = 12$. It came out 12, but the strange thing is that this is true for all of them.* [Students: How does it work, teacher?]
>
> $T6_1$: *It is the same when you do the multiplication. In order to keep this in your mind, we have a bee, the bee will take pollen from the flowers placed in brackets, what does this bee do, first it lands on the 1st flower, then on the 2nd flower, takes honey, then we add the pollen it takes, we subtract, we see that the same result comes out.* 🐝(✿+ ⚘) *Now let's look at the question. "$8 \times (10 + 3)$" here $8 \times 10 + 8 \times 3 = 80 + 24 = 104$ and let's do the operation the other way. Add 10 and 3. 13, multiply by 8. The result is 104. The results do not change. This is what we call the property of multiplication being distributed over addition and subtraction. The number outside, the bee, lands on every flower inside, and our bee can enter from the right. Even if it enters from the right, it still has to start from the left.* [Students: Teacher, but if $8 \times (3 - 10)$, does it start from 10.]
>
> $T6_1$: *No. Is it 3-10 or is it 10 from 3 or not?* [Students: If my teacher does the opposite of what you say, for example, if he multiplies $(10 - 6) \times 2$ by 6 first]
>
> $T6_1$: *But you can't do that, $6 \times 2 = 12$, $2 \times 10 = 20$ How will 20 come out of 12? "$8 \times (12 + 5)$" who is our bee here 8. Put them on the flowers one by one. $8 \times 12 + 8 \times 5 = 96 + 40 = 136$ ... "$(13 + 8) \times 5$" in this operation you can also put 5 on the head, why? Because multiplication does not have the property of changing? Yes, how does it change? For example, $5 \times 4 = 20$ and $4 \times 5$ is 20. You can change it, right? You can think of it as if it was at the beginning $5 \times (13 + 8) = 5 \times 13 + 5 \times 8 = 105$... Here, it is definitely a more practical method to first do the inside of the parenthesis and multiply it with the outside, but if we do not know the dispersion property that we will use here, there will be questions that we cannot do or we will do much simpler operations using the dispersion property. How? Let's take $12 \times 14$ for example. Now look, I will do this multiplication mentally, I will do it without moving pen and paper...* [Students: Teacher, I found the answer 164]
>
> $T6_1$: *Tell me how did you find it?* [Students: Teacher, I multiplied 14 and 12 in order $4 \times 2$ ...]
>
> $T6_1$: *But you forgot a number. What will happen? Watch carefully. I will multiply 12 by 14, but I will multiply 12 easily. Can I write 14 as $10 + 4$? Yes. Now it becomes $12 \times (10 + 4)$. Now I think if I multiply 12 by 10, I have 120 pockets. If I multiply 12 by 4, I have 48 pockets. $120 + 48 = 168$. Can the answer be something else, for example, how about $13 \times 19$?* [Students: $13 \times (10 + 9)$]
>
> $T6_1$: *Suppose I multiply 10 by 9. It is easy to multiply 13 by 10, but it is difficult with 9...* [Student 1: Teacher 11 by 8]

> T$6_1$: *No, let it be easy multiplication...* [Student 2: For example, let's multiply 10 by 13 and then multiply 10 by 13 again and then subtract 13]
> T$6_1$: Oh, *do you understand?* [Student 3: Let's do 20 – 1]
> T$6_1$: *The logic he says is as follows. Let's write 13 × (20 – 1). Let's multiply 13 by 20, 260. I multiplied 13 by 1, 13. 260 – 13 = 247. One missing 19. So I can either add or subtract, whichever is closer.* [Student 4: What if I do 12 × 20?]...

T$6_1$ mathematics teacher provided a lesson environment where questions were flying in the air and provided many feedbacks for her students to reason about the properties related to numbers and operations. Teacher T$6_2$, on the other hand, tried to explain the LO$6_3$ outcome related to the property of distributive property by breaking up the sum. For example, the teacher tried to explain the distributive property of multiplication over addition with a mathematical expression such as $8 \times 37 = 8 \times (30 + 7) = (8 \times 30) + (8 \times 7) = 240 + 56 = 296$. When the teacher was asked why he/she did this, the teacher stated that he/she made a connection with the method of decomposing numbers (or sums), which is related to the mental addition process emphasized since primary school levels. In this sense, the teacher did a rule-oriented teaching with a method that he knew directly without giving the students a chance.

As a result, the data obtained as a result of observing teachers teaching at different teaching levels in the context of generalizing the basic properties of arithmetic are summarized in Table 2.

**Table 2.** Data Obtained in the Context of Generalizing Basic Properties in Arithmetic

| Teacher Codes | Learning Outcomes Codes-Names | No appropriate activity/practice observed | An appropriate activity/practice observed | |
|---|---|---|---|---|
| | | | Teacher-centered | Student-centered |
| T$3_1$ | LO$3_{10}$ – Makes addition with at most three-digit numbers with and without eliminations. | | * | |
| T$3_2$ | LO$3_{11}$ – Shows that changing the order of addition with three natural numbers does not change the result. | | * | |
| T$4_1$ | LO$4_7$ – Makes addition with at most four-digit natural numbers. | | * | |
| | LO$4_{16}$ – Shows that changing the order of multiplication with three natural numbers does not change the result. Examples with parentheses are also included in the operations. | | * | |
| T$4_2$ | | | * | |
| T$5_1$ | LO$5_{10}$ – Determines and uses the appropriate strategy in mental multiplication and division with natural numbers. | | * | |
| T$5_2$ | - | * | | |
| T$6_1$ | LO$6_3$ – Performs operations to apply the common factor bracketing and dispersion property in natural numbers. | | | * |
| T$6_2$ | | | * | |

When Table 2 is examined, it is understood that except for T$5_2$, most of the teachers, both primary and secondary school mathematics teachers, carried out teacher-centered teaching activities related to generalizing the basic properties of numbers and operations, which generally included the traditional approach, and they did not benefit from student-centered activities such as making assumptions or predictions, noticing, verifying or justifying, proving, using physical or virtual manipulatives or multiple representations. Teacher T$6_1$ adopted a student-centered approach in the development of algebraic thinking. In the literature, it has been emphasized that rather than having students memorize rules and properties, teachers should provide opportunities for students to analyze many specific or particular situations that help them go beyond thinking about multiple examples of the thinking underlying mathematical generalizations (Beatty & Bruce, 2012). However, most of the observed teachers did not provide such opportunities for their students. Indeed, Denmana and Leitlez (1988) emphasized the importance of recognizing and generalizing patterns among numbers in order to have a perfect understanding of procedural rules (change, combination, distribution, inverse and order of operations). These are transitional topics from arithmetic to algebra and are necessary for solving algebraic equations.

### 3.2. Activities Performed by Teachers in the Context of Generalizing Relationships Other Than Basic Properties in Arithmetic

The data obtained from the teachers in the context of generalizing relations other than the basic properties in arithmetic were obtained as a result of observing the classroom environments in which the learning outcomes coded LO$3_8$, LO$3_9$, LO$4_{12}$, LO$4_{17}$, LO$5_5$ were addressed. Most of these generalizations involve focusing on relationships rather than numerical calculations, and such generalizations are related to relationships in number classes and the results of calculations. For example, these data include properties that express relationships in number classes or the results of calculations such as odd/even numbers and understanding how these properties can be used in algebraic expression transformations (Bastable & Schifter, 2008).

While explaining learning outcome $LO3_8$, which is related to odd and even numbers, teacher $T3_1$ told the students that odd numbers are 1, 3, 5, 7, 9... and even numbers are 2, 4, 6, 8... and then, in order to make the students understand the situation better, he had a certain number of students stand up on the board and make them pair up and explained the situation to them as follows: "*So, we are saying that when they are paired with each other, it is even, but when they are paired, if one of them remains odd, it is an odd number*."

> $T3_1$: *Now, children, we will talk about odd and even natural numbers. We will look at the ones digit of the numbers. We will call the numbers with 1, 3, 5, 7 and 9 in the ones digit as odd numbers. We will call the numbers with 0, 2, 4, 6, 8 in the ones digit even numbers. Now you may think like this, how do we know whether an odd number in 1 and an odd number in 6 are odd or even?*[Students: How?]
>
> $T3_1$: *Let me explain it to you like this...* [The teacher put 3 students on the board]. *Let's play a game in pairs, let's pair up and find a partner for yourselves. Play against each other.* [One student stays single]. *I put 3 students on the board, but Defne was left alone. Defne has no partner*. [The teacher puts another student on the board]. *Now there are 4 people. Defne also has a pair. This is what I mean by even and odd, let's count 2, 4. So we say that when they pair with each other, it's even, but if one of them remains odd when they pair, it's an odd number. Look, let's take one more person, we have 5 people. One of them remained odd, but when we took one more person, everyone became even.* [In this way, he continued by taking a few more students to the board]. *So one odd two even three odd four even... In other words, we look at the ones digit to find out whether the numbers are even or odd, we call it even or odd accordingly. When there are 1,3,5,7,9, we will call it an odd number and one will always be left out when it is paired, but when there are 0,2,4,6,8, we will call it even and when it is paired, there is no number left out... It is enough to look at the ones digit to determine whether a natural number is odd or even. The sum or difference of an even number is always an even number. The sum or difference of an odd number and an even number always gives us an odd number* [Then the teacher made simple examples].

As can be understood from the dialogue above, this in-class activity concretized the situation in the students about the properties of even and odd numbers and enabled them to comprehend the subject. However, the teacher did not engage in an activity that would enable students to make a discourse verbally or with words in order to generalize the properties of even and odd numbers. In this sense, it would be a better approach in terms of the development of algebraic thinking for the teacher to create an environment that will enable the students to reach a generalization as a result of the activity of making pairs. Teacher $T3_2$ gave the rule about addition of odd and even numbers directly to the students in the context of outcome $LO3_9$. However, the curriculum requires students to discover and express whether the sums of odd and even numbers will be even or odd by using models (MEB, 2018). In this sense, the teacher's direct presentation without giving students the opportunity is not a very appropriate activity in the context of generalizing arithmetic properties other than the basic properties in particular and the development of algebraic thinking in general. As a matter of fact, it is important for the development of algebraic thinking that students start thinking about the properties of numbers rather than focusing on the results of calculations with specific numbers (e.g. 2 + 2 = 4) (Akkan, 2016). In this sense, teachers should ensure that students recognize regularities in the results of addition with odd and even numbers through algebraic reasoning, rather than thinking about different combinations or results (odd + odd = even, even + even = even, odd + even = odd), that is, students should think about individual examples of sums of odd and even numbers. If, as a result of these activities, students understand the pattern for which the sum of any two pairs of odd and even numbers is valid and can identify this pattern, they will have made a generalization. For example, the statement "The sum of an even number and an odd number is always odd" is a mathematical generalization because students have captured a real relationship in a data set (the set of integers). In this way, students can characterize the result of the sum of any even number and any odd number. In addition, teachers can further develop students' understanding of the sum of odd and even numbers by using pictures or concrete materials. By considering the numbers and symbols corresponding to these pictures or concrete materials, they can show that the sum of an even number and an odd number is always odd and that the sum of two even numbers or two odd numbers is even. For example, as in $(2m + 1) + (2n + 1) = 2m + 2n + 1 + 1 = 2 (m + n) + 2$. In this context, none of the teachers carried out an activity in which students could make and test conjectures, nor did they design an environment enriched with multiple representations or physical and virtual manipulatives.

In the context of outcome $LO4_{12}$, teacher $T4_1$ did not give students the opportunity to make assumptions or reasoning, but explained the subject by directly giving a rule with the sentence "*...10 is counted backwards from the diminishing number*". However, it should be kept in mind that students can develop different strategies (for example, they can first subtract 90 and add 20, or see that zeros are ineffective and subtract 7 from 9, etc.) and reach generalizations on their own.

> $T4_1$: *Let's write mental subtraction with 10. When subtracting mentally with 10 and multiples of it, the number subtracted from...* [Students: What is the number subtracted? Isn't it the larger number?]
>
> $T4_1$: *… Sorry, 10 is counted backwards from the subtracted number. For example, let's subtract 70 from 590. 590-580-570-570-560-550-540-540-530-520...Let's see how many 10s we counted, we counted*

> *7. What is 7 tens? 70. Let's see if we get the same result.* [He makes them do subtraction by writing one under the other].

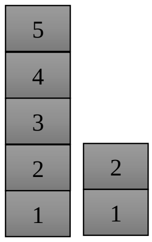
Similarly, we can see the rule-oriented teaching approach of teacher $T4_1$ in the lesson environment including outcome $LO4_{17}$. The dialog of the teacher's lesson environment including $LO4_{17}$ outcome is as follows:

> $T4_1$: *What is the result of $A \times B$ in the operation $45 \times A = 45000$ and $B \times 100 = 23000$?*
> $T4_1$: *What is the subject of multiplication here?* [Students: Multiplication with 10, 100 and 1000.]
> $T4_1$: *Let's find the number A and then find the number B. He multiplied 45 by a number and found 45000. If I multiply 45 by 45, what will be 45000?* [Students: 1000]
> $T4_1$: *Actually, the answer shows me 3 zeros. What was the number with three zeros?* [Students: 1000]
> $T4_1$: *He multiplied the number B by 100 and found 23000. I will use this to find this number B. What will I do? I will remove the two zeros here. I take it back because I multiplied it by 100. What I have left is 230. A =1000, B= 230, 1000×230=? I don't involve the zeros at all. 23×1=23 How many zeros are there? 4. Then 230000.*

Classroom teacher $T4_2$, on the other hand, had the students write a title as "*multiplication in a short way*" and then tried to explain the relations of the current situation without allowing the students to notice and generalize them, and even without entering into any dialogue with the students. Teacher $T4_2$ said, "*For example, let's multiply the number 25 by 10. We will not take zero into account here. We will multiply one and twenty-five. Then we will add the zero to the back side.*"

It is understood from the dialogue below that the mathematics teacher $T5_1$ did not engage in any activity that would allow students to notice regularities or relationships related to mental addition and subtraction in the lesson environment involving $LO5_5$.

> $T5_1$: *Mental addition has certain rules. We can separate tens and ones when adding natural numbers. For example, if I ask you to add 34 + 25 mentally, you can add 30 and 20 together and get 50. You can add 4 of 30 and 5 of 20 together and get 9. You add 50 and 9 together, 59. Add 12 and 28 together...* [Students: 8, 2 and 10. You have one.]
> $T5_1$: *No, it is not like that, you will add 2 and 8 together and keep it in your mind. You will add 10 and 20.* [Student: 40, teacher, 2 and 8 are 10. The sum of 10 and 20 is 30. The sum of 30 and 10 is 40].
> $T5_1$: *Another strategy is to add to 10 or 20. For example, let's take 38+53. Now, look at 38. What happens if I add 2 to 38? Let's add 40 and 53, 93. Let's subtract the 2 I added before from 93, 91. Let's do another example. 8 + 79 = ? I'll do it like this. I'll add one to 79. What's that, 80. 80, 8 more 88. I added 1, now subtract it, 87... Separating tens and ones. Let's first subtract 33 from 94. Let's subtract 30 from 90, 60. Let's subtract 3 from 4, 1. 60, 1 more, 61.*
> $T5_1$: *Now let's look at the question in the book. What do we mean by completing the number to the missing. Firstly, friends, I explain from the simple and give the rule.* [Teacher drew a shape on the blackboard.]




> $T5_1$: *If we subtract 5 from 2, does the 3 above remain? Well, if we subtract 2 from 5, 3 remains. Then how many do we need to get 2 to 5?* [Students: 3]
> $T5_1$: *Let's give another easy example. 25 – 12. See, now I will take 12 and add it with something and try to reach 25, okay? Let's add 12 and 10, 22. How many times can I add 22 to get 25? I got 10 + 3 = 13.*

In this lesson, which was taught with the traditional teaching style, the teacher did not take any action that would allow the students to develop an appropriate strategy or reason about addition. However, as can be understood from the outcome statement "*Determine and use strategies in mental addition and subtraction with two-digit natural numbers*", it is recommended in the curriculum that students develop strategies by making use of existing relationships and use the strategies they develop in different situations.

Similarly, in the lesson environment containing outcome $LO5_5$, the classroom teacher $T5_2$ carried out an activity as follows to help students find the method that would allow them to recognize regularities or relationships related to mental addition with the help of the smart board.

> $T5_2$: *... For example, we did shopping. We need to make a calculation, right? How much will we take, how much will we give? So what we need to do is to do mental addition and subtraction. This means that we need mental addition and subtraction at every moment of life. Let's see how we do it. Do you have a practical method?* [Students: Rounding]
> $T5_2$: *Rounding gives approximate results. For example, let's give an example, you will add 23 and 35… You add three and five separately, you add twenty and thirty separately, right?* [Students: Yes]
> $T5_2$: *Is there anyone else who can tell me another method?* [Student: Teacher, I do the numbers directly from my mind. I write them one under the other and add them]
> $T5_2$: *That is super. Let's see what other methods there are. Let's watch...* [The teacher opened a video on the smart board. In the video, there were operations with the numbers 55 and 23] *Yes... By dividing into tens and ones. So, how do we do it? So, we take 50 + 5 + 20 + 3 tens as one. The tens are 70,*

*the ones are 8, the total is 78. When I wrote it like this, it seemed difficult, but it becomes easier in your mind, right? … Then let's write it down in our notebook right away. Let's say the first method is to separate tens and ones.* [Students were asked to solve a few more examples by asking different questions]… *Yes, our second method is to write counting on it, friends.* [The teacher turned on the video again on the smart board]... *We got it, right... Let's give an example then. Let's first understand what we are doing. We keep the larger number in our minds, add the other number ten by ten and add the remaining one last. Let's give an example, let our example be 48 + 35. What do we do? Which one is bigger? 48. We kept 48, come here, you. What do we do with 37? I break it down ten by ten. 48 ten more 58, ten more 68, ten more 68, ten more 78. 7 more, 85. Yes, what happened is that we broke it into minced meat and then added it to this. Friends, these are different methods. Adding 40 and 30 together and adding 8 and 7 together. Separating the tens from the ones and counting on this. Does everyone make their own example in their notebooks?* [Students: Yes]

$T5_2$: *Yes, let's see what the third method is. Yes, the method is to start with numbers that are easy to add. Now what do we add easily? We add tens and hundreds easily, right? If we come across a 10 or a 100, we add them easily. So, if there are numbers in the environment that add up to a hundred or that add up to a hundred, what are we going to do with them? Do you understand? We will say that we will add the easy ones first and add the others afterwards.* [The teacher opened the video on the smart board and solved the example of 16 + 25 + 34. Then he continued the video]...*I will move on to the other method now. There was one more method that was not included here. Then let's do it like this, call it Method 4. Addition with reference to 10. So listen to me like this. Again, like before, we can add more easily when the end is zero, right? So first of all, what are we going to do? For example, let's take 37 + 16. If we add how many to 37, how many zeros do we make it zero?* [Student: 3]

$T5_2$: *I need a 3 next to 37. Then I will take 3 from 37 and give it to the other one. Yes, this time it is not 37, what happened? 40. Now we say 40 + 13, what is the total? 53...* [The teacher gave a few more examples]… *Yes, we move on to subtraction now. Like the methods of addition, this time we subtract. Yes, let's call it subtraction. Subtraction from the mind. First, subtraction by separating tens and ones. Let's see, I think you can do this. We will subtract 23 from 48. Separating the tens, we'll subtract 20 from 40 and 3 from 8. What happens is 20 out of 40 is 20. 3 out of 8 is 5. The answer is 25... Yes, it comes up a lot in the questions, ten by ten subtraction, let you know. It is used a lot in subtraction. The second method is onar, onar subtraction.* [The teacher opened the 45 – 22 operation on the video on the board.]... *How do we subtract on and on. Before 45, we first subtract ten and what's left? 35. Subtract it again and what's left? 25... There is one 2 left. We subtract 2 from 25, 23.*

At the beginning of the lesson, $T5_2$ teacher tried to motivate the students by attracting their attention to the subject and to ensure that the students comprehend the outcome with the question-answer method by making use of technology. The teacher tried to make students comprehend different methods with more than one example. However, the teacher did not engage in activities for students to find relationships and draw a general conclusion by developing different strategies.

In the lessons related to outcome $LO6_6$, mathematics teacher $T6_1$ engaged in activities that would provide students with the opportunity to make their own rules or generalizations by observing the relationships between numbers. For example, the dialog between the teacher and his/her students about creating divisibility rules for 2 and 3 is as follows.

$T6_1$: *The rule of divisibility by 2. Now when we say the rule of divisibility by two, can you give me an example.* [Student 1: 2 pairs; Student 2: 2, 4, 6, 8, 10, 12, 14, ...]

$T6_1$: *Now, the numbers your second friend said have a common feature. Friends, how can we create a rule?* [Student: All numbers divided by two are even numbers]

$T6_1$: *So even numbers are divided by two without remainder. Now let's continue, friends, what can be the remainder of a number divided by two* [Student: 0, 2].

$T6_1$: *If it is 2, does the division process end?* [Student: no, it can be 1 and it can be 3]

$T6_1$: *If it is 3, we continue the division* [Student: 0 or 1].

$T6_1$: *Yes, 0 or 1, friends, when we divide something, the remainder always has to be what it should be, it has to be less than the divisor number, if we divide by two, the remainder is either 0 or 1, whatever we divide it by has to be at least 1 less... Let's move on to the rule of divisibility by 3. Now let's write the numbers that are multiples of three and establish a relationship. Numbers divisible by 3 are 3, 6, 9, 12, 15, 18, 21 ,... Is there a relationship between these numbers? Who will tell us?* [Student: Half even and half odd]

$T6_1$: *Then I can't say that even can be divided and odd can't be divided, right here* [Student: Some even and some odd can be divided]?

$T6_1$: *But how will we know, for example, I told you a very big number, like 1 million. How will we know whether this number is divided by 3 or not?* [Student: We can divide it by easy division]

$T6_1$: *You say so... For example, let's say 120, 129 and 150. Aren't they all multiples of 3? Yes. But how do we decide when the number gets bigger.* [Students: By division ... by subtraction, teacher]

$T6_1$: *How do we subtract* [Student: For example .... does not exist, let's multiply it]

> $T6_1$: *Let's multiply what by what, anyway, let's take a clue and look at the sum of the numbers. What is the sum of the digits of 120, 129 and 150 respectively? What kind of relationship is there?* [Student: 3, 12 and 6. Teacher: 3 and a multiple of 3. So we will add the numbers and divide the sum by 3]
>
> $T6_1$: *Then what is the rule?* [Student: If the sum of the digits is 3 and multiples of 3, it is divided by three without remainder. Wouldn't it be wrong if it is remaindered?]
>
> $T6_1$: *No, it would not be wrong, the remainder of the quotient is the remainder of that number divided by 3. For example, let's give an example, let's divide 1205 by 3. Now, we start from the sum of the digits. 1 + 2 + 0 + 5 = 8. Is 8 exactly divisible by 3? Or what is the remainder?* [Students: No, it is not divisible by 3, the remainder is 2. ] *Then the remainder of 1205 divided by 3 is 2.* [The teacher tried to explain the rules of divisibility by 4, 5, 6 and 9 similar to the rules for 2 and 3].

Similarly, teacher $T6_1$ carried out activities to help students discover relationships and form rules by using relationships in relation to the rules of divisibility by 4, 5, 6, 9. In particular, they allowed students to make predictions and make assumptions. In the lesson related to the outcome coded $LO6_6$, mathematics teacher $T6_2$ said, "*If the ones digit of a number is 0 2 4 6 and 8, that is, if it is even, that number can be divided by 2 without remainder...*", "*A rule related to the ones digit friends, numbers with ones digit 0 or 5 can be divided by 5 without remainder...*". " and "*To find out whether a number is divided by 3 or not, you add the digits of that number, and if it is exactly divisible by 3, then that number is exactly divisible by 3...*" directly without giving the students a chance to speak. Then the teacher tried to explain the rules of divisibility in a very complex way by giving different numbers and explaining the remainders of these numbers divided by 2, 3, 4, 5, 6 and 9. In this sense, the teacher did not allow the students to see the relationships and express the rules in their own sentences, but tried to make the students understand the rules by making long explanations herself.

As a result, the data obtained as a result of observing teachers teaching at different teaching levels in the context of arithmetic generalizations other than basic properties are summarized in Table 3.

**Tablo 3.** Data Obtained in the Context of Generalizations Other Than Basic Properties in Arithmetic

| Teacher Codes | Learning Outcomes Codes-Names | No appropriate activity/practice observed | An appropriate activity/practice observed | |
|---|---|---|---|---|
| | | | Teacher-centered | Student-centered |
| $T3_1$ | $LO3_8$ – Understands odd and even natural numbers. Real objects are used when working with odd and even natural numbers. | | * | |
| $T3_2$ | $LO3_9$ – Expresses whether the sums are odd or even by examining the sums of odd and even natural numbers on the model. | | * | |
| $T4_1$ | $LO4_{12}$ – Mentally subtracts two-digit natural numbers that are multiples of 10 and three-digit natural numbers that are multiples of 100 from three-digit natural numbers. | | * | |
| $T4_1$ | $LO4_{17}$ – Multiplies at most three-digit natural numbers by 10, 100 and at most nine multiples of 1000; multiplies at most two-digit natural numbers by 5, 25 and 50 in a short way. | | * | |
| $T4_2$ | | | * | |
| $T5_1$ | $LO5_5$ – Identifies and uses strategies for mental addition and subtraction with two-digit natural numbers. | | * | |
| $T5_2$ | | | | * |
| $T6_1$ | $LO6_6$ – Explains and uses the rules of divisibility by 2, 3, 4, 5, 6, 9 and 10 without remainders. | | | * |
| $T6_2$ | | | * | |

When Table 3 is examined, it is observed that $T5_2$ and $T6_1$ teachers tried to support algebraic thinking with student-centered activities in the process of forming rules about divisibility, while the other teachers tried to support algebraic thinking with teacher-centered activities. Therefore, in the context of generalizing relations other than the basic properties in arithmetic, it is understood that teachers do not engage in enough activities/actions for the development of algebraic thinking. Fujii and Stephens (2008) defined semivariational thinking as the ability of students to use general explanations of why a numerical expression such as 78 – 49 + 49 + 49 = 78 is true and then use specific examples that can be seen as a general relationship (a – b + b = a). Britt and Irwin (2011) pointed out the importance of students using the three-stage semiotic system of "*numbers as semivariables; words; literal symbols* (p. 154)" respectively to express any generalization and stated that providing opportunities for students to be aware of different generalizations is an important activity for algebraic thinking. From Britt and Irwin's (2011) perspective, let us look again at the expression 48 + 25 – 25 = 48. Students may realize that subtracting and adding 25 does not change the result and that the equation is true even if any number is substituted for 25. Next, students can realize that adding and subtracting the same number from any number does not change the result. Finally, assuming that the expression is true for any number, students can write 78 + b – b = 78 and generalize a + b – b = a from this expression. In such generalizations, basic properties (b – b = 0 and a + 0 = a) can be used a lot. For example, the understanding that the order of numbers does not change the result of multiplication of two numbers can enable students to produce more flexible solutions for multiplication. Students' thinking of the expression $16 \times \frac{1}{2}$ instead of $\frac{1}{2} \times 16$ can help them understand that $16 \times \frac{1}{2}$ is half of 16.

### 3.3. Activities Performed by Teachers in the Context of Inverse Operations

The data obtained from the teachers were obtained as a result of observing the classroom environments where $LO4_7$, $LO4_{25}$, $LO5_5$, $LO5_{12}$, $LO6_4$ objectives were addressed. Since inverse operations have an important role in the mathematical modeling process or in the abstraction of algebraic elements such as equality, equivalence, equation, inequality, which are the products of the modeling process (Kieran, 2004), curricula emphasize inverse operations (subtraction is the inverse operation of addition and vice versa, division is the inverse operation of multiplication and vice versa) starting from the first grade of primary school. In our country's primary school mathematics curriculum, the outcome involving the relationship between addition and subtraction operations is given at the 2nd grade level, while the outcome involving the relationship between multiplication and division operations is given at the 4th grade level (MEB, 2018). In this sense, there is no acquisition at the 3rd grade level that includes operations that are the inverse of each other. As a matter of fact, it was determined from the classroom observations of $T3_1$ and $T3_2$ that both teachers did not emphasize the relationship between addition and subtraction.

However, while teaching addition in natural numbers, teacher $T4_1$ explained finding the ungiven sum by giving a rule as follows: "*When finding the ungiven sum... we subtract from the sum…*".

> $T4_1$: *One of the operations we will do while finding the ungiven sum is to subtract from the sum while finding the ungiven sum... For example, if I do one example... Let's take ? + 28 = 103. "First collected + Second collected = Total". This means that when we do not know one of the summands, we subtract one of the given summands from the sum and find the other summands. So 103 – 28 = ? Does 3 add up to 8? No. I went to the neighbor and the neighbor didn't have it either. I went all the way to the other neighbor. I took a tenner. That's 0 here and 10 there. Three out of eight or not. I went again. This time there were 10 tens. I took one. There are 9 tens left. That's 13. Does 8 come out of 13?* [Students: Yes]
>
> $T4_1$: *I subtracted 8 from 13, 5 from 5, 2 from 9, 7. The result is 75. We write 75 here with our red pen. Also, sometimes children may not give us the sums when they give us the sums. Sometimes they do not ask for the first or second addition, sometimes they may ask like this. For example, let's look at the following addition together... 5∆3 + 291 = □34... Now when you look here, 3, 1 more 4 and there is no hand with a square (He writes the operations one by one under each one.) △ + 9 = 3 cannot be. Then what should this 3 be 13. Then I count on 9, 9, 10, 11, 12, 13 what did he do?* [Students: 4]
>
> $T4_1$: *It means that 4 will come where the square is. When we forget the one at hand, we find the operation wrong. Then our result 1 + 5 + 2 = 8 is equal to 5 in the square and 8 in the triangle...*

According to the dialog, the teacher provided information directly and did not design a classroom environment that would allow students to develop different strategies and reasoning. However, the teacher included examples of placeholders such as 5∆3 + 291 = □34 in addition to the standard addition examples, which is especially important for the development of algebraic thinking. While teaching outcome $LO4_{25}$, teacher $T4_1$ gave the rule directly to the students as follows.

> $T4_1$: *Division is the opposite of multiplication. Because in multiplication, we multiply or multiply. In division, on the other hand, we divide and share equally. Now let's think like this. I have 10 liras, I continued earning 10 liras every day for a month. Did I multiply it? I multiplied it... Now I divided this 300 liras I earned to my five children. Did it decrease? It did. Therefore, division is the opposite of multiplication. Because of this feature, multiplication is used to check the correctness of a division operation. Shall I show you how? (Quotient × Divisor) + Remainder = Divisor" should give the number... For example: 324 ÷ 6 = 54. Is this operation right or wrong? I multiply the quotient by the divisor 54 × 6 = 324. Then my operation is correct.*

As can be understood from the dialogue, it was determined that the $T4_1$ classroom teacher did not engage in activities or practices that would allow her students to recognize and generalize the relationships between inverse operations. Then, in order to reinforce the rule, the teacher tried to show the correctness of the generalization by giving more than one example to the students.

Teacher $T4_2$, just like teacher $T4_1$, presented the relationship between multiplication and division directly without giving students the opportunity to reason. For example, in response to the question "What can be the opposite of division?", the students gave different answers such as multiplication, subtraction and addition, and then, without giving the students a chance, the teacher coded $T4_2$ said, "*Multiplication... I mean, the opposite of division is multiplication. The opposite of addition is subtraction. ... then we will do multiplication...*" directly expressing the rule or relationship.

As can be understood in the dialog below, while explaining learning outcome $LO5_{12}$, mathematics teacher $T5_1$ started the lesson with a statement such as "*Friends, multiplication and division are twins of each other just* *as addition and subtraction are twins of each other.*” He wrote the properties consisting of four items related to the subject in the students' notebooks.

> $T5_1$: *Friends, just as addition and subtraction are twins, multiplication and division are also twins.* [After saying the above statement, the teacher had the students write four rules as follows...] *(1) In multiplication, the unknown product is found by division. For example, to find the square 23 × □ = 345, 345 ÷ 23 = 15. (2) If the divisor cannot be given in a division operation, the divisor is multiplied by the quotient. If there is a remainder, it is added. For example, the divisor is 12 and the quotient is 7. What should we do? We multiply. 12 × 7 = 84. What is the maximum number of remainders?* [Student: 3]. *(3) If the divisor is not given in division with remainder, the divisor is divided by the quotient again. For example, take 516 ÷ □ = 23. The remainder is 10. Why are we dividing? I explain: We have 516 walnuts. I put 23 walnuts in each plate. How many plates do I need? How many walnuts are left over? How do you find it* [Student 1: By dividing; Student 2: Will the remainder stay like that, teacher?] *It will be 22 and the remaining walnuts will be 10. (4) In division by remainders, if the quotient is not given, the divisor is divided by the divisor...*

This situation shows that the teacher memorized the relationship between multiplication and division as well as the elements such as multiplier, divisor and divisor. What is expected from $T5_1$ mathematics teacher is that students discover the relationship between these two operations and generalize the four given properties by finding the relationships themselves.

$T5_2$ stated that in the context of outcome $LO5_5$, the mathematics teacher told one of the students that one of the additions could be found by trial and error to find the one not given in an addition operation. However, the teacher asked all of the students, “*Well, can we find it with a more practical method, for example, subtraction?*” and the students stated that they were not used to this kind of method. On the other hand, it was observed that the mathematics teacher $T5_2$ tried to arouse curiosity in the students while explaining that addition and subtraction are inverse operations in a different lesson environment, encouraged her students to think, and therefore, many different opinions or voices emerged from the students.

> $T5_2$: *Now let's come to finding the ungiven in subtraction. Yes, friends, in the subtraction process, we wrote in the previous lesson, didn't we, subtracting, subtracting, difference? ... Now, friends, how will we find the ungiven subtraction, subtraction, difference? Let's see.* [The teacher demonstrated the concepts of less, subtract, difference on subtraction]. *Now, friends, I will explain it in a simple way so that we can understand it better. If there are 9 subtracted and 5 subtracted, the difference is 4. I'm telling you that I couldn't find the subtractor, what is the subtractor? Can you find the subtractor? What can you do?* [Students: By subtracting 4 from 9]
>
> $T5_2$: *We find the subtraction by subtracting 4 from 9. We subtract the difference from the subtraction. You can remind yourself with a simple operation to remember this, okay? For example, how we were doing it... I mean, you can remember it with a simple operation. Let's write it down right now. We subtract the difference from the subtracted amount to find the subtracted amount. I didn't give the subtraction, what do we do?* [Students: Addition, we add the difference with the subtractor].
>
> $T5_2$: *We add the difference with the subtraction and find the subtraction. Let's write that down right away. We add the difference with the subtractor to find the missing subtraction. Here, subtraction is actually reversed, right?* [Students: Yes]
>
> $T5_2$: It *is like the provision of the work. We know what provisioning is, don't we?* [Students: Yes... No...]
>
> $T5_2$: *Are there still people who haven't heard of provisioning?* [Students: None]
>
> $T5_2$: *Isn't it like checking the correctness of that operation?* [Students: Yes]
>
> $T5_2$: *For example, we subtracted 5 from 9, 4. We add 4 and 5 again, 9. Then what we do is that we find what is missing. Sometimes we may encounter different situations. For example, how will the result change if the deficit is increased by this much? How will the result change if the difference is increased by this much? Questions arise. Have there ever been any. What happens if the deficit is increased, friends, let's look at the example again. Let's say let's increase the deficit by one, it became 10 and 5 came out of 10. Look, I increased the deficit by one and the difference increased by one. Then, if the deficit is increased, the difference increases by that much. What happens if the subtractor is increased? Shall we look at our example again? I increase the subtracted, this time 9, I increased the subtracted by one, 5. What is the difference?* [Students: 4]
>
> $T5_2$: *The more the output increases, the more the difference?* [Students: Decreasing]

According to the dialogue above, even though the students did not reach the correct result and did not realize the inverse operation themselves, they were given the opportunity to think. Because the teacher presented the generalization as “*In addition, if we do not know one of the addends, we subtract the other addend from the sum.*” The mathematics teacher $T5_2$ adopted a similar approach to the addition process while teaching the subtraction process.

However, in the observations made with the sixth grade mathematics teachers, it was observed that the teachers did not exhibit any activity involving inverse operations, even in the lesson in which the $LO6_4$ outcome related to the backward working (inverse operation) algorithm, which develops in parallel with the inverse operations, was taught. However, developing students' own problem solving (informal solution) strategies plays an important role in the transition to formal solution methods (French, 2002). Baroody (1998) emphasized that the backward working (reverse operation algorithm) informal solution strategy that students use while solving verbal problems is more effective than formal solution strategies. The backward working strategy is frequently used in the teaching and learning of problem solving and mathematical proof within the scope of elementary and advanced mathematics programs (Bayazıt & Aksoy, 2009), and the use of operations that are inverse of each other in this solution strategy - subtraction is the inverse operation of addition and the inverse is also true, division is the inverse operation of multiplication and the inverse is also true - is important (Akkan, 2016). In this context, it is important for teachers to design activities in which students explore the relationship between inverse operations. In particular, students can learn the relationships between inverse operations throughout their learning about “Fact Families” activities. When we think algebraically about a relationship between two numbers, we think of the first number as changing and transforming into another number. For example, we can think of the expression 3 + 6 = 9 both as the sum of two parts (3 and 6) to obtain a whole (9) and as changing from 3 to 9 by adding 6. With the reverse operation, i.e. subtracting 6 from 9, we can come back to the starting number (3). Teachers can also design environments where students can more easily understand the relationships between inverse operations by using unordered table values (Warren & Cooper, 2008). Finally, it is evident that this understanding of inverse operations can help with the two activities that Kieran (2004) identifies in the development of algebraic thinking, namely (1) focusing on relationships, not just on the calculation of a numerical answer, and (2) focusing on the inverse of operations, not just on the operations themselves (the idea of thinking about doing or undoing an operation). Therefore, students need to understand operations as well as their inverses.

In conclusion, Table 4 summarizes the data obtained from observing teachers teaching at different levels of education in the context of the relationships between inverse operations.

**Tablo 4.** Data Obtained in the Context of Relationships between Inverse Operations

| Teacher Codes | Learning Outcomes Codes-Names | No appropriate activity/practice observed | An appropriate activity/practice observed | |
|---|---|---|---|---|
| | | | Teacher-centered | Student-centered |
| $T3_1$ | - | * | * | |
| $T3_2$ | | * | * | |
| $T4_1$ | $LO4_{25}$ – Recognizes the relationship between multiplication and division. | | * | |
| | $LO4_7$ – Performs addition with natural numbers with at most four digits. | | * | |
| $T4_2$ | | | * | |
| $T5_1$ | $LO5_{12}$ – Understands the relationship between multiplication and division and finds the elements not given in the operations (multiplier, quotient or divisor). | | * | |
| $T5_2$ | $LO5_5$ – Identifies and uses strategies for mental addition and subtraction with two-digit natural numbers. | | | * |
| $T6_1$ | $LO6_4$ – Solves and constructs problems that require four operations with natural numbers. | | * | |
| $T6_2$ | | | * | |

When Table 4 is examined, it is understood that except for teacher $T5_2$, both primary and secondary school mathematics teachers carried out teacher-centered teaching activities involving the traditional approach in exploring the relationships between inverse operations, while teachers $T3_1$ and $T3_2$ did not engage in activities to support algebraic thinking.

## 4. Conclusions

Most of the teachers did not allow students to discover and express the basic properties of number systems (change, merger, dispersion, etc.) in their own language by making use of the relationships between numbers and operations; rather, they engaged in teacher-centered approaches or activities. Teachers did not create environments that would enable students to focus on the relationships between numbers and operations rather than the results of numbers and operations and to generalize these relationships. Teachers mostly gave the formulas or rules related to these basic properties directly to the students in their lectures, that is, they told the generalization by using a single example and only the relationship between two numbers, but they did not design teaching activities in which students could reach the generalization by using two or more examples. However, rather than having students memorize rules and properties, teachers need to provide opportunities for students to analyze many specific or particular situations that help them go beyond thinking about multiple examples to thinking about the thinking underlying mathematical generalizations. In addition, informal interviews revealed that teachers believe that students cannot reach generalizations on their own. The data obtained from the teachers in the context of generalizing the basic properties of arithmetic were obtained as a result of lesson observations involving the objectives related to this component ($LO3_{10}$, $LO3_{11}$, $LO4_{7}$, $LO4_{16}$, $LO5_{10}$ and $LO6_{3}$), and activities related to this component were not observed in lesson environments involving different objectives. Most of the objectives related to this component involve the generalization of arithmetic relations that include the basic properties of numbers and operations, i.e. the relations obtained as a result of observing how operations behave and how they are related to each other, and reasoning about these generalizations. As a result, it was understood that most of the mathematics teachers in both primary and secondary schools carried out teacher-centered teaching activities related to generalizing the basic properties of numbers and operations, which generally included the traditional approach, and did not benefit from student-centered activities such as making assumptions or predictions, noticing, verifying or justifying, proving, making use of physical or virtual manipulatives or multiple representations.

Most of the teachers did not engage in activities that involved generalizing relationships other than the basic properties of arithmetic (odd/even numbers, mental addition, divisibility rules, etc.) and mostly used teacher-centered approaches. Teachers did not engage in activities that involved different combinations or results related to the sums of odd and even numbers, i.e., activities that would enable students to recognize regularities in the results of addition with odd and even numbers through algebraic reasoning rather than thinking about individual examples of sums of odd and even numbers. Similarly, teachers did not engage in activities that would allow students to develop an appropriate strategy or reason about addition (i.e., identify and use strategies in mental addition and subtraction with two-digit natural numbers). The data obtained from the teachers in the context of generalizing relations other than the basic properties in arithmetic were obtained as a result of observing the classroom environments in which the learning outcomes coded $LO3_{8}$, $LO3_{9}$, $LO4_{12}$, $LO4_{17}$ and $LO5_{5}$ were addressed. Most of these objectives involve focusing on relationships rather than numerical calculations, and such generalizations are related to the relationships in number classes and the results of calculations. In addition, teachers did not use pictures or concrete materials to develop students' understanding of the sum of odd and even numbers. However, by considering the numbers and symbols corresponding to the pictures or concrete materials, they can show that the sum of an even number and an odd number is always odd and that the sum of two even numbers or two odd numbers is even. As a result, it was understood that teachers did not engage in enough activities/actions for the development of algebraic thinking in the context of generalizing relations other than the basic properties in arithmetic.

It was found that most of the teachers did not engage in activities or practices that would allow their students to recognize and generalize the relationships between operations that are inverses of each other. The fourth grade teacher memorized the relationship between multiplication and division as well as the elements such as multiplier, divisor and divisor. Then, in order to reinforce the rule, the teacher tried to show the correctness of the generalization by giving more than one example to the students. This process is not very compatible with the development activities of algebraic thinking in the context of generalized arithmetic. Some of the fifth grade mathematics teachers, on the other hand, tried to arouse students' curiosity while explaining that addition and subtraction are inverse operations in a different lesson environment, encouraged students to think, and therefore, it was observed that many different opinions or voices emerged from the students. Although the students did not reach the correct conclusion and did not realize the inverse operation themselves, they were given the opportunity to think. However, in the observations made with sixth grade mathematics teachers, it was observed that the teachers did not exhibit any activities involving inverse operations, even in the lesson in which the acquisition related to the algorithm of working backwards (inverse operation), which develops in parallel with the inverse operations, was taught. The data obtained from the teachers in the context of inverse operations were obtained as a result of observing the classroom environments in which the objectives $LO4_{7}$, $LO4_{25}$, $LO5_{5}$, $LO5_{12}$, and $LO6_{4}$ were taught. In our country's primary school mathematics curriculum, the outcome involving the relationship between addition and subtraction is given at the 2nd grade level, while the outcome involving the relationship between multiplication and division is given at the 4th grade level. In this sense, there is no acquisition at the 3rd grade level that includes operations that are the inverse of each other. As a result, it was understood that both primary and secondary school mathematics teachers, with the exception of $T5_{2}$, carried out teacher-centered teaching activities involving the traditional approach in exploring the relationships between inverse operations, and it was observed that third grade teachers did not engage in activities to support algebraic thinking.

## 5. Recommendations

As a result of the research, it was determined that secondary mathematics and classroom teachers give very little space to activities that support algebraic thinking in the context of generalised arithmetic. In this context, in-service training programmes that include activities or practices to develop algebraic thinking skills can be organised for teachers. In addition, curricula and outcomes can be re-examined and new outcomes and explanations that support early algebraic thinking can be added to the curricula and existing outcomes can be revised. In line with the recommendations made in the literature regarding the activities of early algebraic thinking, studies can also be conducted to examine the practices developed or designed by both classroom and mathematics teachers and the effects of these practices.

In the context of generalized arithmetic, as students gain experience with the four operations involving numbers, they may begin to notice certain patterns or regularities in the properties of these four operations. For example, the focus on generalizing the property of addition in multiplication is a form of algebraic thinking. Students may initially learn to express the addition property using their natural language (e.g., changing the grouping of factors does not change the product, or multiplying three natural numbers does not change the result if any two of the factors are multiplied first, etc.). As students become mathematically mature and proficient, they can learn to express these ideas in more formal ways, using symbols to represent any three numbers: $(a \times b) \times c = a \times (b \times c)$ for each real number. Models are also an important tool for proving the unification property and for understanding the conceptual knowledge underlying such mathematical generalizations.

However, suppose that students are given a problem situation as "$67 + 83 = \square + 82$" (Carpenter et al., 2003). It is natural for students who solve this problem only with numerical calculations to reach the result. However, the students who realized that 82 was one less than 83 and wrote 68 in the box focused on relationships rather than numerical calculations. Students who perform such effective numerical manipulations can develop algebraic thinking skills by making sense of both the algebraic properties behind the equation ($a + b = \_ + (b - 1)$ or $\_ = a + 1$) and the (different) uses of variables ($61 + x - x = 61$, etc.).

In addition, students should understand operations as well as their inverses. However, it is obvious that students understand the inverse relationship between addition and subtraction more easily, while it takes longer to understand the inverse relationship between multiplication and division. In this sense, students can learn the relationships between inverse operations in the learning processes related to "Fact Families" activities.

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