Mental Calculation Strategies as a ‘Missing Link’ between Arithmetic and Algebra – Insights into the 'Auxiliary Task' and its Role in the ‘Cognitive Gap’

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Abstract: Mental calculation strategies are important arithmetical operations in primary school mathematics lessons, but a proceptual understanding of highly relational strategies such as the ‘Auxiliary Task’ might also play an important role in the emergence of pre-algebraic thinking processes as prior insights from this study show. Following upon these indications, n = 18 learners’ proceptual understanding of the mental calculation strategy ‘Auxiliary Task’ is analysed in this article with a Design-based Research study utilizing interpretative analyses of learning processes. The analyses show that the fostering of a proceptual understanding of the ‘Auxiliary Task’ might lead to a) pre-algebraic generalizations of the numbers and a creative ‘analytical noticing’ and b) specific ‘Grundvorstellungen’ – thus multi-facetted mental models – might emerge. Especially two ‘Grundvorstellungen’ seem to be of high relevance in such a pre-algebraic thinking of the ‘Auxiliary Task’: The understanding of mathematical structures behind the ‘Auxiliary Task’, which is the ‘compensation strategy’, as well as a notion of numbers-as-indeterminate. The latter is an important aspect in the transition from primary to secondary school since it precedes an understanding of variables-as-indeterminate, ‘bridging’ an arithmetic and algebraic understanding of numbers.

Keywords: Pre-algebra, Auxiliary Task, Compensation rule, Design-based Research, Interpretative analyses

1. Introduction: Mental calculation strategies, the ‘Auxiliary Task’ and the ‘cognitive gap’

In mathematics education, an important shift in the perception of mental calculation strategies has happened in the last decades: Instead of being regarded as ‘calculation strategies’ only, recent studies highlight the importance of a *conceptually reflected, proceptual understanding* of mental calculation strategies (Britt & Irwin 2008; Serrazina & Rodrigues 2021; Kuzu 2022b). However, in Germany as well as in most other countries worldwide, mental calculation strategies are (still) being taught in most schools as solely calculation-oriented approaches with little or no reflection of the conceptual facets of the strategies (see Kuzu & Nührenbörger 2021; Kuzu 2022a). For example, when learning the so-called HTU-Strategy, meaning a successive calculation of hundreds (H), tens (T) and units (U), students might only learn how to conduct the different calculation steps without visualizing the meaning of the steps with enactive or iconic objects, thus without a reflection on the conceptual side of the strategies (see Selter 2001).

This is problematic or rather a ‘lost potential’ from at least three perspectives: First, a reflection of the conceptual meaning of mathematical objects and procedures leads to *more consolidated learning processes* through the ability to think in larger coherent theories or conceptual networks (see section 2.1.; Freudenthal 1976; Gray & Tall 1992; Tabach, Hershkowitz & Schwarz 2006); second, a viable understanding and *generalization* of mathematical rules such as the compensation rule depends on a viable conceptual understanding of specific strategies such as the ‘Auxiliary Task’[1] (see section 2.2 and 2.3) and third, specific aspects such as the carryover might be taught better by using adequate manipulatives for visualizing the *meaning* of it, which might help prevent arithmetical impairment and mathematical learning difficulties (Gaidoschik 2019). Although these aspects show the importance of teaching mental calculation strategies in a conceptually reflected way, there are still open questions with regard to the general question on *how* to utilize different approaches and *which* strategies might be particularly important for reflecting specific forms of mathematical knowledge, e.g. a viable understanding of mathematical rules: The HTU-Strategy is a stepwise and rather direct strategy, which utilizes a more algorithmic way of calculating mentally, whereas the ‘Auxiliary Task’ is a complex strategy utilizing the compensation rule and demanding a preception of specific properties of numbers, e.g. the approximity to next tens, hundreds etc. or to complements (see Kuzu 2022b). There are diverse cardinal as well as ordinal possibilities (Kuzu 2022a) to explain the ‘Auxiliary Task’, but Kuzu & Nührenberger (2021) also show that *explaining* the conceptual meaning of the ’Auxiliary Task’ is a highly challenging process for learners – especially with regard to the verbalization of the conceptual meaning behind each step – and needs a scaffolding of languaging-processes, meaning the utilization and learning of language means with a cognitive function (see Swain 2006; see section 3.2) beside of a good visualization through manipulatives and objects, or

[1] The ‘Auxiliary Task’ – a term being coined in the German tradition and focussing more the modification of one number instead of multiple numbers (see Selter 2001) – is also known as compensation strategy in international works.

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rather accompanying the utilization of manipulatives in the sense of a content-and-language-integrated approach (see Pöhler & Prediger 2015; Kuzu 2019; Wessel 2020).

In this article, especially the 'Auxiliary Task' will be focussed because of a subject didactical necessity to gain insights into learners' interpretation and generalization of it (see Kuzu 2022b): The necessity comes from missing insights into learners' proceptual interpretations of the 'Auxiliary Task', but prior studies gave first insights into a particular importance of the strategy for overcoming the 'cognitive gap' between 'arithmetic' and 'algebra' (see Herscovics & Linchevski 1994; see section 2.1.), meaning unknown processes in the transition: For example, learners might develop a first notion of variables-as-unknowns when working with boxes hiding a number of objects with regard to complements etc. (see Steinweg, Akinwunmi & Lenz 2018). For figural numbers, word problems, functional tasks, box-tasks etc., the importance is examined well in the last years but for more arithmetic approaches – like the 'Auxiliary Task' – , there are only a few studies, mostly with a focus on equation-like tasks in primary school (e.g., Schwarzkopf, Nührenbörger & Mayer 2018).

2. Theoretical background: Mathematical knowledge, pre-algebra and the 'Auxiliary Task'

An analysis of mental calculation strategies such as the 'Auxiliary Task' needs insights into at least two important facets: First, what happens *mentally* when thinking about numbers and calculation procedures (see section 2.1), especially if these are conceptually reflected and generalized (see section 2.2.), and second, what subject matter knowledge is necessary to conduct the mental procedure behind the strategy (see section 2.3.).

2.1. Prescriptive and descriptive dimensions and the macro- vs. micro-level of mathematical knowledge

The conceptual understanding of mathematical objects is not an isolated form of knowledge: 'Grundvorstellungen' of mathematical objects – a term being rooted in the German idealism as well as the New Education Movement ('Reformpädagogik') through the works of Pestalozzi, Herbart, Kühnel etc. and meaning prototypical mental models of mathematical concepts and procedures (see vom Hofe & Blum 2016; Hefendehl-Hebeker, vom Hofe, Büchter, Humenberger, Schulz & Wartha 2019; Kollhoff 2022) – are highly important prescriptive models for learning processes, for example when learning fraction concepts such as *part-of-whole* or *part-of-multiple-wholes* (see Tunç-Pekkan 2015; Glade & Prediger 2017). There are three important aspects to 'Grundvorstellungen' (see vom Hofe & Blum 2016; Kollhoff 2022):

1. The development of *adequate mental representations* of a mathematical concept or procedure (e.g., a cardinal thinking of 'addition' as the adding of one or more amounts to other amounts)
2. The connection *to mentally represented, prototypical activities* (e.g., an automatized imagination of a merging-activity when thinking about 'addition')
3. The application as *'linking bridges' between everyday situations and mathematized situations* (e.g., the thinking of 'addition' when collecting or gathering objects).

Thus, 'Grundvorstellungen' are *multi-facetted and highly adaptable mental models* – they have to be thought of as 'ideals' or objectives of what has to be taught for a fully viable understanding – and it is not only important to know *how to think* a mathematical object, but also *how to construct it* with an activity and where to use it, that is *in which situations*. The first facet is the 'core' of a 'Grundvorstellung', which has to be developed in a viable way, and the other two aspects do support this process in reciprocal way: Conducting *mentally represented, prototypical activities* (facet 2) and *utilizing a specific understanding of a mathematical object or process in an everyday situation* (facet 3) needs as well as leads to *adequate mental representations* (facet 1) (see Kollhoff 2022). Thus, 'Grundvorstellungen' are a complex form of knowledge in a network-form, but they can be, or rather have to be linked with other mathematical concepts in form of a broader knowledge-network – meaning larger coherent conceptual theories going beyond single 'Grundvorstellungen' (see Gray & Tall 1992; Tabach, Hershkowitz & Schwarz 2006) – and for constructing these broader knowledge-networks, transfer-processes are highly important, meaning a process where learners *"connect their prior learning to new situations and applications and in this way extend and further develop their understanding of the concepts they are learning"* (Kollhoff 2022, p. 51). A good example for such a transfer process is the necessary connection of a conceptual understanding of fractions to a conceptual understanding of percentages (see Pöhler & Prediger 2015).

'Grundvorstellungen' and transfer processes, however, are not the only aspects, a further analysis of mathematical objects shows a complexity going beyond it: In-between concepts and procedures, there are so-called 'procepts', which are understood as *"[…] an amalgam of both process and concept […] the manifestation of the process which can itself be manipulated as a mental object"* (Gray & Tall 1992, p. 210). Gray & Tall (1992) emphasize the necessity to differentiate between 'process' and 'procedure' by defining both as distinct knowledge-forms, with 'process' meaning the general intention to be carried out – for example the addition or subtraction process – and 'procedure' meaning the particular method used by an individual at a given time – for example an (unreflected) mechanical action and algorithmic routine for conducting an addition or subtraction (see Gray & Tall 1992). They highlight that for successful learning of mathematics, processes and not only procedures have to be focussed since those who only learn procedures will show *“success in being able to perform the current task, but* [an] *inability to coordinate the processes into any larger coherent theory”* (Gray & Tall 1992, p. 213). As a step beyond focussing ‘processes’, they emphasize a *reflected* understanding of processes, thus the ‘procepts’: From a prescriptive perspective, learners should understand the meaning of a process by thinking about it in a conceptual way, e.g., with cardinal or ordinal manipulatives or representations, *before* automatizing it (see Gray & Tall 1992).

Besides these two prescriptive aspects, the *individual notions* of learners with regard to specific ‘Grundvorstellungen’, for example individual notions in interpreting a share, are also of high relevance (see Glade & Prediger 2017; Prediger, Kuzu, Schüler-Meyer & Wagner 2019), which is the descriptive dimension. In a constructivistic sense, *individual notions* are understood as *relational mental models* or *schemes* by which the individuals can capture the meaning of mathematical objects and phenomena in everyday- or innermathematical-situations (see Fischbein 1989; Kuzu 2019) and interindividual differences in interpreting mathematical objects are seen as typical for learning processes (see Piaget 1977; von Glaserfeld 1991), making it necessary to design adequate learning environments with the potential to stimulate ‘cognitive conflicts’ in case of non-viable individual notions (see Waxer & Morton 2012). Thus, descriptive insights into learners’ interpretations might be an important *starting point* for an analysis of learning processes since highly individual conceptual nuances can typically be reconstructed in every learning process, for example when interpreting the direction or sequence of thinking the ‘part’ in the ‘whole’ (see Kuzu & Prediger 2017; Prediger et al. 2019). These individual notions of learners’ may not be fully compatible with ‘Grundvorstellungen’, making it necessary to reconstruct the learners’ individual notions – ranging from non-viable to partially-viable, intuitive concepts – and to give them opportunities to re-shape them to viable concepts (see Fischbein 1975; Prediger 2008; Schneider, Vamvakoussi & Van Dooren 2012). By analysing learners’ individual notions with such a bottom-up empirical approach, ‘chances’ and ‘hurdles’ can be focussed more adequately but this does not mean that a top-down, objective oriented structuring of learning process is irrelevant. Rather, a combination of both perspectives is important for a profound analysis of learning process following the key question ‘*What should the learners ideally learn* and *which individual notions of concepts* are there as starting points to be refined or re-shaped?” (see Zwetzschler 2015; Kuzu 2019).

The prescriptive and descriptive dimension including ‘Grundvorstellungen’, transfer processes, ‘procepts’ and individual notions might be regarded as the *micro-level* of learning mathematical objects, but then there is the much broader perspective, the development of knowledge in mathematical topics such as ‘arithmetic’ and ‘algebra’ (see section 1) in the sense of the ‘spiral principle’ (Bruner 1960), which could be regarded as the *macro-level*. The ‘spiral principle’, going back to Bruner’s (1960) early psychological works, refers to different levels in the development of knowledge and comprises broader topics such as ‘decimal system’, ‘measurement’, ‘arithmetic’ or ‘algebra’ etc. (see Wittmann 2021). The main idea is that these topics should not only occur *once* in the learning process but multiple times and on different levels by utilizing learners’ prior knowledge and continuing it into further levels (see Büchter 2014; see Wittmann 2021). In this sense, every mathematical topic can be thematized in a simplified way prior to the ‘fully’ schematized and formalized way and is linked to further topics, matching the idea of knowledge networks (see Bruner 1960; Glade & Prediger 2017; Wittmann 2021). A good example would be the topic of ‘measurement’: Fracturing a whole into equal parts and focussing a specific part of it is a core activity and an important conceptual facet when conceptually reflecting on the structure of measures, e.g., of ‘length’. By developing such an understanding, learners also develop a first notion of the divisibility of numbers, which is an important pre-concept to the ‘part-of-whole’ concept for fractions (see Tunç-Pekkan 2015; Kuzu 2019). Furthermore, if centimeter squares (squares with an edge length of 1cm) are used to restructure or fill out an object, a first notion of ‘area’ might emerge in learners’ minds prior to the usage of square units. Thus, between different ‘levels’ of mathematical topics, there are specific ‘bridges’ or prior forms of knowledge. Although grounding works with regard to the ‘spiral principle’ date back to the 1960s, still, there are not yet enough insights into how these ‘bridges’ between topics are built but especially a viable, adaptable understanding of mathematical objects – thus the development of viable ‘Grundvorstellungen’ within these topics – seems to have an important function in this process (see vom Hofe & Blum 2016; Hefendehl-Hebeker et al. 2019; Wittmann 2021). To summarize the interplay of ‘Grundvorstellungen’, individual notions and topics, the connection between these three aspects can be thought of as a complex network (see figure 1).

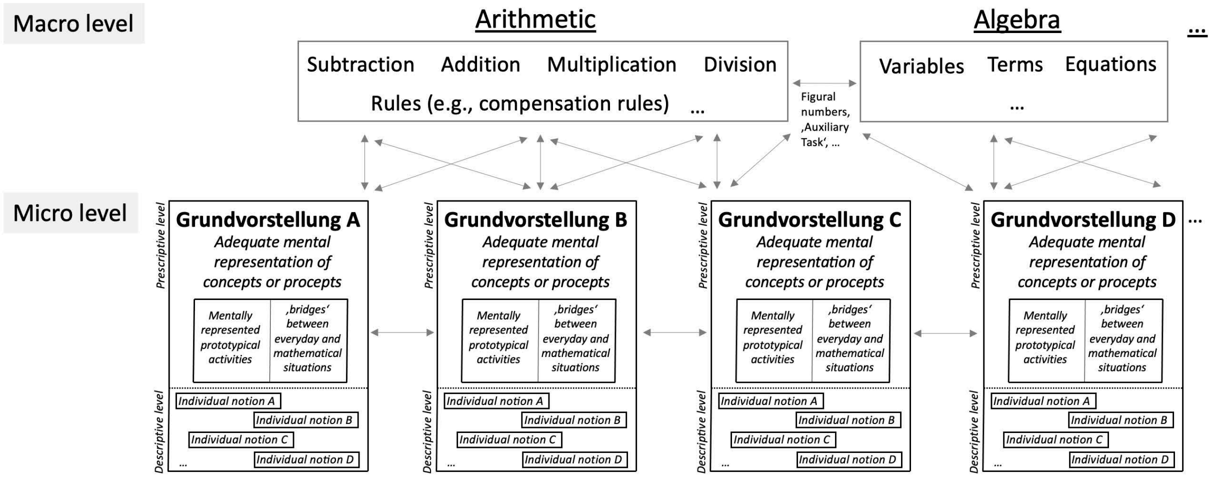


Figure 1. The interplay of 'Grundvorstellungen', individual notions and topics (Author's own elaboration)

As it is visible in figure 1, the macro level consists of broader topics like 'arithmetic' and 'algebra' and there are multiple processes and objects per each topic. Each process and object itself comprises a plurality of interwoven 'Grundvorstellungen' – for the process of subtraction for example, there is the 'Grundvorstellung' of 'taking-away', 'comparing states', 'comparing changes' and 'complementing, inverse task' (see Hefendehl-Hebeker et al. 2019) – , and these 'Grundvorstellungen' consist of the three facets a) adequate mental representation, b) mentally represented prototypical activities and c) 'bridges' between everyday and mathematical situations on the prescriptive level as well as of learners' individual notions on the descriptive level. Moreover, these 'Grundvorstellungen' are linked with each other too: The 'Grundvorstellung' of 'complementing, inverse task' for subtraction for example is an additive notion and thus highly connected to 'Grundvorstellungen' of addition. Between the broader topics, there are specific tasks and formats, for example, figural numbers, mental calculation strategies like the 'Auxiliary Task' etc., which connect different 'Grundvorstellungen'. Precisely with regard to the latter – the connection between 'arithmetic' and 'algebra' through the 'Auxiliary Task' – further empirical insights into students' interpretations and into involved 'Grundvorstellungen' are missing (see Kuzu 2022b).

2.2. Pre-algebraic generalizations in the context of the 'Auxiliary Task'

Since especially between the two highly important topics 'arithmetic' and 'algebra' insights into transitional processes are missing, Herscovics & Linchevski called it the so-called 'cognitive gap' (see Herscovics & Linchevski 1994; see section 2.1). In the last years, the terms pre-algebra or synonymously early algebra emerged from a theoretical perspective to describe this specific 'gap' (see Steinweg 2013; Kieran, Pang, Schifter & Ng 2016; Kieran 2018; Radford 2018). Steinweg (2013) defines pre-algebraic thinking by referring to the notion of 'procepts' (see section 2.1.) as the conceptual understanding of *"relations, patterns, and structures of concrete numbers, mathematical equations and terms"* (p. 12-13, translation from author). This definition emphasizes not only a focus on *reflected processes* (see section 2.1), but also on the way this is done in primary school: Through the use of concrete numbers, terms and equations, not yet necessarily with alphanumerical symbols (see Steinweg 2013; Radford 2018).

In the context of the 'Auxiliary Task' this means that learners do have to reflect the 'procept' behind the strategy (see section 2.3), which means a generalized understanding of the process through a reflection of the compensation process by using manipulatives and representations (see Kuzu 2022b). Generalization' is understood as a comprehension of properties and relations of mathematical objects going beyond specific cases (see Steinbring 2006). Prior analyses show that at least three aspects are important for fostering a pre-algebraic understanding of the 'Auxiliary Task' effectively:

- *Language sensitiveness:* The 'Auxiliary Task' is highly complex since the learners have to think and verbalize specific steps that are interwoven. They have to see the property of numbers (e.g., the approximity to the next tens or the possibility to use tens-complements) and then they have to modify the focussed number (either the first or second) in a first step, followed by a compensation in a second step, if they use the asynchronous variant of the 'Auxiliary Task' (for the second variant, the synchronous variant, they would have to modify and compensate in one single step). Prototypically, learners use sentences like *"One has to take away … first"* and *"We have to put back/ take back another… now/ after that, because we took away too much/ added too much…"* when they try to explain the steps for the first time (see Kuzu & Nührenbörger 2021). Thus, they have to sequence their thinking with adequate language means. Furthermore, specific language means like the 'group language' might be needed (see Götze & Baiker 2021) since in the context of multiplication and division, learners have to describe how a 'bundle' of objects is added or taken away (see section 2.3)

- *Usage of adequate manipulatives:* For fostering a conceptual reflection, adequate manipulatives and representations should be used. The learners have to see what happens when modifying the numbers and compensating it afterwards, e.g., by adding a specific amount of discrete of objects to 'fill' the gap until the next tens and by taking away the same amount in the last compensation step, and there is a high plurality of possible manipulatives a teacher could use, from ordinal to cardinal manipulatives, the latter being differentiable in discrete, continuous or mixed forms (see Kuzu 2022a). Most tasks in the context of the 'Auxiliary Task' are symbolic and if manipulatives are used, then mostly ordinal manipulatives are used and only very few tasks and studies do utilize a cardinal approach for explaining the conceptual facet of the 'Auxiliary Task' (see Britt & Irwin 2011; Kuzu & Nührenbörger 2021). Yet, cardinal manipulatives are important because they make visible what would be invisible or implicit otherwise: the first modification step. In an ordinal representation, the first 'jump' includes the modification, it is 'hidden' within the jump, but when using cardinal manipulatives, learners have to add or take way the amount they need to modify the first or second number, thus it is visible more directly (see Kuzu 2022a; see section 2.3.).
- *A conceptual reflection of the 'Auxiliary Task' through an adequate design of the learning environment:* Conceptually reflecting the 'Auxiliary Task' and generalizing its meaning is a complex learning goal, which can only be achieved if an adequate learning environment is designed. A learning environment with a succession of design elements following the three main design principles 1) *Fostering of a richly entwined conceptual understanding preceding procedural calculation*, 2) *Content-and-language-integration through register relation* and 3) *Sequencing tasks with the aim of fostering generalization processes* seems to foster pre-algebraic generalizations in a successful way (see section 3.1; Kuzu 2022b).

Further research does substantiate these three aspects. For example, Steinweg (2019) also emphasizes such an approach within the ReCoDE-model that consists of the steps Recognise-Continue-Describe-Explain (see Steinweg 2019; Akinwunmi & Steinweg 2022) – especially with regard to pattern-recognition – and Radford (2018) similarly describes how pre-algebraic thinking emerges through repeated and reflected arithmetic operations on patterns (see Radford 2018). Both approaches strongly refer to pattern recognition, but as Schwarzkopf, Nührenbörger & Mayer (2019) state, a similar potential lies in reflected arithmetical operations since *"arithmetical knowledge in primary classes already includes abilities of conversion that ultimately harbor algebraic potential [...] without relying on formal algebraic tools such as elaborated representations and terms"* (Schwarzkopf, Nührenbörger & Mayer 2018, p. 195).

2.3. The proceptual facets of the 'Auxiliary Task'

The 'Auxiliary Task' is one of several mental calculation strategies. What makes the 'Auxiliary Task' exceptional – or rather unusual with regard to classroom-norms – is the fact that learners do something which is normally not allowed in mathematics lessons: To *change* the numbers as one wants. Using the 'Auxiliary Task', such a manipulation of the numbers in a first step is allowed, if the emerging inequality is balanced out in subsequent steps. This means, as Threllfall (2002) explicates, that learners have to develop a specific form of 'analytical thinking': They have to *see* the possibility to change the numbers before starting to calculate, for example the approximity of the given numbers to the next tens, hundreds etc. (see Threllfall 2002; Padberg & Benz 2011). This is what differs the 'Auxiliary Task' from other strategies like the HTU-strategy (see section 1), where the learners do not need to think about the numbers or the term beforehand, they just can start calculating and it will lead to a solution (see Selter 2001). This might seem like a small difference with regard to the effectiveness of a calculation process, but for the emergence of a pre-algebraic understanding of numbers and terms, it is a crucial difference (see section 2.1; Kuzu 2022b).

There are several possibilities to construct an 'Auxiliary Task', thus a task which makes calculation easier because of e.g., rounded-up or rounded-down numbers: The learners could modify the task – for example a term with two numbers – by changing both numbers at the same time, which would be the *synchronous* variant of the 'Auxiliary Task'. When calculating 332- 118, learners could add +2 to the second number 118 and at the same time add +2 to the first number[2], which is a modification and compensation in one single step. Another possibility, being more adequate for introducing learners into the 'Auxiliary Task', would be the option to change one number in a first step and to compensate in a second step, which is the *asynchronous* variant of the 'Auxiliary Task' (see figure 2)

[2] Sometimes, this simultaneous strategy is also called 'Balancing strategy' since learners do compensate directly.

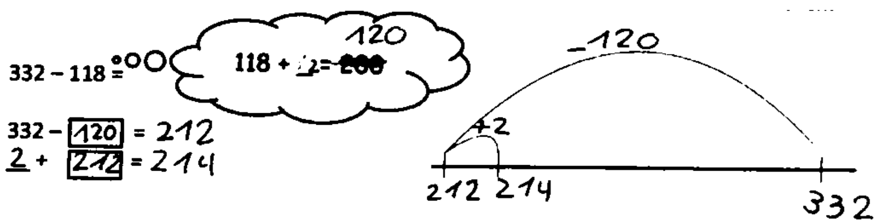


Figure 2. The ‘Auxiliary Task’ explained with an ordinal number-line and a thinking bubble (Kuzu 2022b)

In such a stepwise process, learners might keep a better overview on the process of compensation, which is crucial since from a proceptual perspective, what learners do when constructing an ‘Auxiliary Task’ is to utilize the so-called *compensation rule*, which is the mathematical structure behind the ‘Auxiliary Task’ (see Lüken & Akinwunmi 2021): Every numerical modification is allowed if an adequate compensation is done. This might sound like a simple rule, but a structural understanding of the compensation rule means that learners have to understand what happens if one rounds up, rounds down, changes the number by taking away or adding a specific amount and when modifying an addition, subtraction, multiplication or division task, which is fundamentally different: For constructing an ‘Auxiliary Task’ in the context of addition and subtraction, one modifies and compensates single elements and for constructing an ‘Auxiliary Task’ in the context of multiplication and division, one modifies and compensates groups of objects (see Kuzu 2022b; Götze & Baiker 2021). To illustrate the difference, a cardinal representation of the proceptual facet might help (see figure 3).

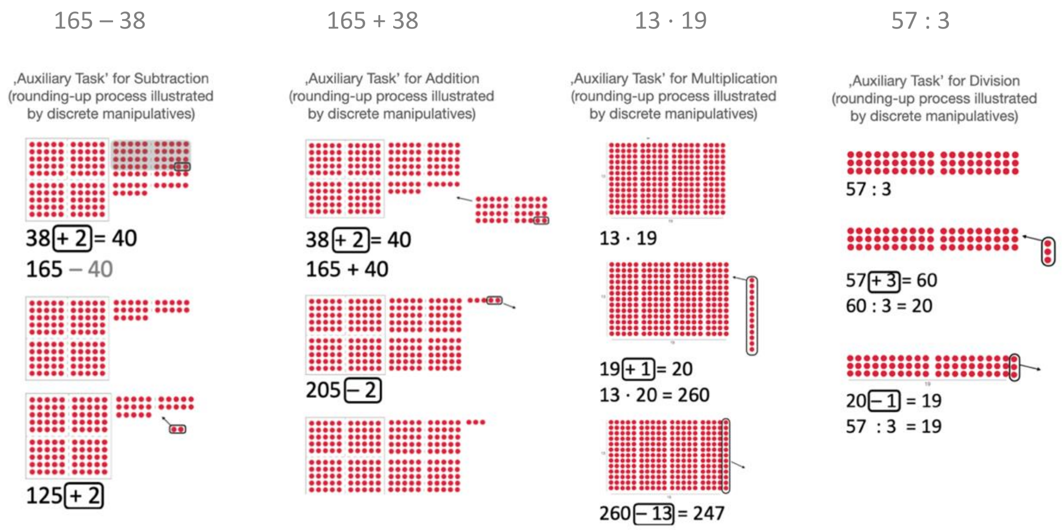


Figure 3. Modification and compensation for rounding up, cardinally illustrated (Kuzu 2022b)

In figure 3, only the process of rounding-up is illustrated proceptually – by showing its conceptual meaning with cardinal manipulatives – , but a fully viable understanding of the compensation strategy would also need an understanding of the rounding-down process as well as an understanding of further modifications, like doubling or halving the numbers, building complements etc. (see Kuzu 2022b).

3. Method: ‘Design-based Research’ with an interpretative ‘Interaction Analysis’ of learning processes

Because of the necessity to develop and evaluate a new and adequate learning environment, the ‘Design-based research’ framework was chosen (see section 3.1) and as analytical framework, the ‘Interaction analysis’ (see Krummheuer & Naujok 1999; see section 3.2) was chosen due to the necessity to gain carefully abducted insights into learners’ individual notions and ‘Grundvorstellungen’ (see section 2.1) being verbalized when explaining and generalizing the ‘Auxiliary Task’ (see section 2.2).

3.1. The research framework: ‘Design-based research’ (Wittmann 1995)

Because adequate learning environments with a differential approach to a fostering of the proceptual meaning of the ‘Auxiliary Task’ as well as insights into learners’ individual notions in the context of such a conceptually reflected understanding of the ‘Auxiliary Task’ are missing, the method used in this study is Design-based Research with an explorative focus on learning processes, thus a mainly qualitative approach. The framework of Design-based Research (see Wittmann 1995; Gravemeijer & Cobb 2006; Prediger, Gravemeijer & Confrey 2015; Nührenbörger, Rösken-Winter, Link, Prediger & Steinweg 2019; Wittmann 2021) means the successively evaluated development of a learning environment and is conducted iteratively, with each iteration consisting of a theoretical and empirical facet. It is a method being used when adequate learning environments are missing or seem to be improvable due to new insights (see figure 4).

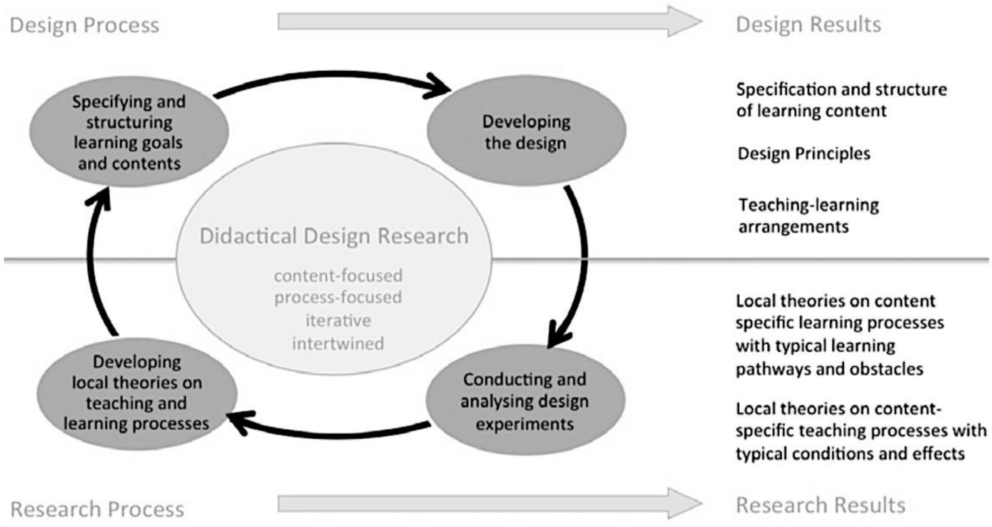


Figure 4. Design-based research cycle (illustration from Prediger & Zwetzschler 2013)

The Design-based Research cycle according to Prediger & Zwetzschler (2013) consists of four central phases (see figure 4): 1. Specifying and structuring of learning goals and contexts, 2. Developing [or adapting] the design, 3. Conducting and analysing design experiments and 4. Developing local theories on teaching and learning processes. These four phases are conducted iteratively and at the end of each iteration, there are adapted design results (design principles and design elements) as well as research results (insights into learning processes). The first iteration starts with a first, not yet optimal design idea (see Prediger & Zwetzschler 2013).

Since there are only very few learning environments fostering a pre-algebraic understanding of the 'Auxiliary Task' by reflecting the proceptual facet of the strategy (see section 2.2 and 2.3) – especially with cardinal manipulatives (see Kuzu 2022a) – , the utilization of a Design-based Research approach seems appropriate. Explorative insights into learners' interpretations of the 'Auxiliary Task' are necessary (see section 1), which is why 'descriptive theory elements' were focussed mainly, meaning the description of *"a certain phenomenon qualitatively or quantitatively"* (see Prediger 2019, p. 7). In case of the possibility to explain those phenomena, 'explanatory theory elements', meaning an identification of backgrounds or causes, were focussed also (see Prediger 2019). The *design elements* (tasks, manipulatives etc.) being used in the learning environment for fostering a proceptual understanding of the 'Auxiliary Task' were developed according to specific *design principles* – understood as maximes being derived from theoretical aspects as well as empirical insights (Van den Akker 1999). The design principles were shortly mentioned in section 2.2. and for further insights and explanations on the development and implementation of the design principles, see Kuzu (2022b). Primarily, the asynchronous variant of the 'Auxiliary Task' is focussed in this study because of its single-step structure, making it more adequate for introducing learners to the strategy and the compensation rule (see section 2.3), but in some sessions also discovered the possibility to use the more 'compact' synchronous variant and that was regarded as a legitimate alternative as long as the learners could reflect the proceptual background of the strategy.

In total, three design cycles were conducted from May 2020 until February 2022 and following the four phases of Design-based Research (see Prediger & Zwetzschler 2013), each cycle consisted of a designing and overworking phase, a conduction phase, an analysis of learning processes and an inference of local theories with regard to teaching and learning processes (see figure 5).

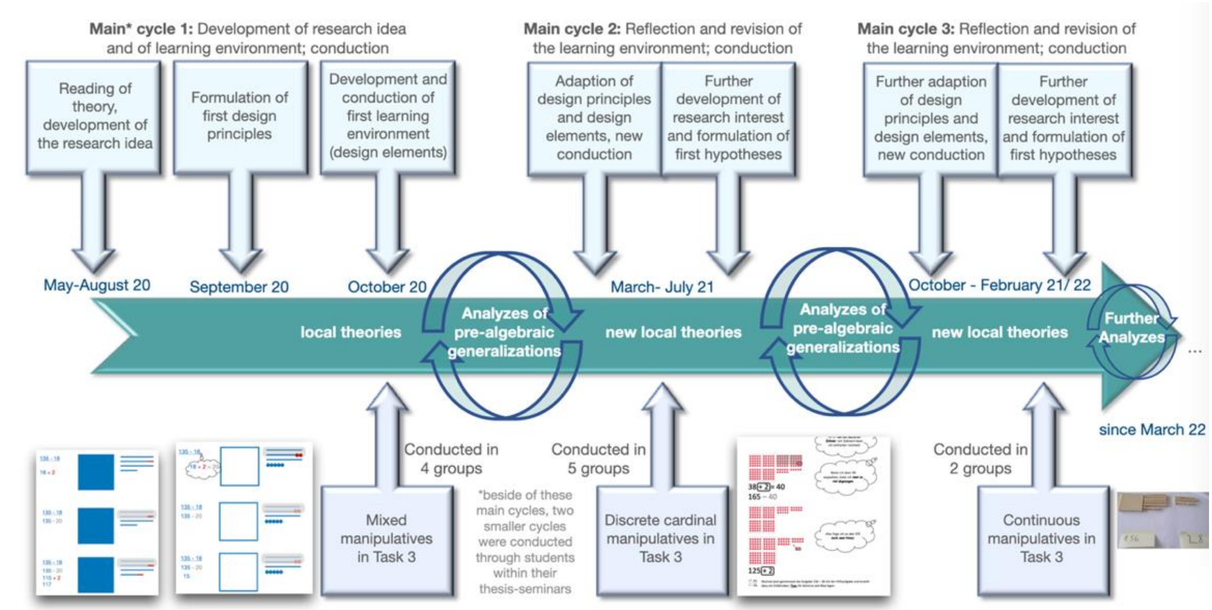


Figure 5. The Design-based Research cycles of the study (see Kuzu 2022b)

From cycle 1 to 3, specific necessities emerged with regard to the teaching and learning process. As described in section 2.2, especially a variation of the cardinal manipulatives (from continuous, mixed to discrete manipulatives) as well as detailed revision of the language means used in the learning environment were necessary. In the last cycle, 'explanation' videos were utilized and additionally, multilingual learners were examined.

3.2. The analytical framework: Interpretative 'Interaction Analysis' (Krummheuer & Naujok 1999) with 'Epistemological Triangles' (Steinbring 2005)

The learning process analyses in-between the Design-based Research cycles were realized by conducting an interpretative turn-by-turn 'Interaction Analysis' (see Krummheuer & Naujok 1999), an analysis method being rooted in three sociological methodologies/ theories and sharing its premises:

- *Ethnomethdology*, which is a methodology focussing social interactions, interaction practices and the construction of meaning through individual and collective actions. It aims at the *„investigation of the rational properties of indexical expressions and other practical actions as contingent ongoing accomplishments of organized artful practices of everyday life"* (Garfinkel 1967, p. 11). Garfinkel (1967) describes these 'indexical expressions' by referring to Husserls phenomelogogical works as expressions with (hidden) properties and latent meaning dimensions, as *"expressions whose sense cannot be decided by an auditor without his necessarily knowing or assuming something about the biography and the purposes of the user of the expression, the circumstances of the utterance, the previous course of the conversation, or the particular relationship of actual or potential interaction that exists between the expressor and the auditor* […] *Time for a temporal indexical expression is relevant to what it names. Similarly, just what region a spatial indexical expression names depends upon the location of its utterance."* (see Garfinkel 1967, p. 4-5). Thus, 'indexicality' means a local, temporal and individual/ personell 'situatedness' of utterances which has to be considered – or rather reconstructed carefully – when analysing social interactions, and especially in everyday interactions, highly indexical expressions are normal: An expression like "I do not like this now" can only be understood or rather de-indexicalized, if one understands the meaning behind "this" and "now".

- *Symbolic interactionism*, which is an interactionistic theory with three premises: *"The first premise is that human beings act toward things on the basis of the meanings that the things have for them. Such things include everything that the human being may note in his world – physical objects, such as trees or chairs; other human beings, such as a mother or a store clerk; categories of human beings, such as friends or enemies; institutions, as a school or a government; guiding ideals, such as individual independence or honesty; activities or others, such as their commands or requests; and such situations as an individual encounters in his daily life. The second premise is that the meaning of such things is derived from, or arises out of, the social interaction that one has with one's fellows. The third premise is that these meanings are handled in, and modified through, an interpretative process used by the person in dealing with the things he encounters."* (Blumer 1969, p. 2). These interpretative 'meaning construction processes' are realized by using language means/ words, gestures or abstract symbols referring to objects, ideas or actions. They are 'symbolic' in that they derive their meaning from the social action involved with them, rather than being inherently connected to or indicative of the things themselves (see Blumer 1969).

- *Objective hermeneutics.* It is a methodology for analysing complex texts, which includes not only written texts but also other meaningful human productions such as films, pictures, and paintings. Its primary objective is to reconstruct latent meanings in a text, which which goes beyond the meaning intended by the author. In order to achieve this, a word-by-word analysis is conducted by formulating extensive interpretations for each word until an analytical saturation point is reached before moving on to the next words (see Oevermann et al., 1987). These interpretations are considered as hypotheses and are expanded, rejected, or differentiated with each word (see Oevermann et al., 1987). The 'Interaction Analysis' focuses on turns instead of individual words due to two reasons: a) the analysis of interactions instead of texts, and b) the relational structure of mathematical concepts and objects, which often becomes visible only through a holistic analysis of individual utterances (see Krummheuer & Naujok, 1999).

The main goal of the interpretative 'Interactional Analysis' is to carefully reconstruct and analyse learners' *individual notions* and interpretations of mathematical objects, processes, rules, norms etc. by formulating hypotheses in a carefully conducted, multi-perspectivistic turn-by-turn analysis, mostly in groups of researchers (see Krummheuer & Naujok 1999; Meyer 2009; Brandt & Tiedemann 2019; Schütte, Friesen & Jung 2019; Kuzu 2022b). Individual notions are understood in a descriptive sense as viable, partially-viable or non-viable 'mental models' learners might develop (see Bauersfeld 1980; Fischbein 1989; Kuzu 2019; see section 2.1).

In this study, the turn-by-turn analysis was conducted in two complementary steps: In a first open step, all turns – with a 'turn' being understood as an interaction-related utterance of an interaction-participant (see Schütte, Friesen & Jung 2019) – were carefully analysed and discussed in groups of researchers with regard to the research interest of this study (learners' interpretations and generalizations of the 'Auxiliary Task'). In this turn-by-turn analysis, hypotheses were abducted in an evidence-based way – meaning with reference to utterances or utterance parts – and then these hypotheses were compared with further hypotheses emerging in the next turns. The turn-by-turn analyses were *interpretative*, meaning an open-mindedness to multiple possible explanations and reasons for specific answers, actions or reactions of learners as well as to a non-explainability, which is the so-called 'interpretative worldview' (see Jungwirth 2003). With regard to a priorly formulated hypothesis, further hypotheses in the next turns could be supplementary/ confirming, conflicting or parallel hypotheses, and if a successive confirmation occurred, so-called '*explanation* hypotheses' were marked (if possible) (see Jungwirth 2003), being understood as *"frequently abducted and intersubjectively plausible hypotheses* […]*"* (Kuzu 2022b, p. 8). In case of the emergence of an explaining hypothesis with regard to mathematical objects or process with relevance the 'Auxiliary Task', a second analysis step, being an in-depths analysis of epistemological process, was conducted by using so-called 'Epistemological Triangles' (see Steinbring 2006; Nührenbörger & Steinbring 2009). These 'Epistemological Triangles' consist of three facets: A sign-related facet, an object-related facet – the 'reference context' – and a conceptual facet (see figure 6).

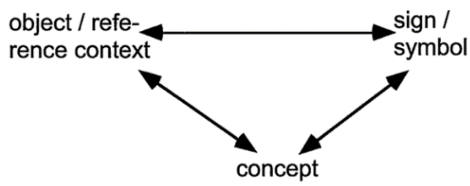


Figure 6. Epistemological triangles (see Steinbring 2005; 2006; Nührenbörger & Steinbring 2009)

In these triangles,

- 'sign' stands for mathematical objects or process, whose interpretation is necessary in an interactional situation (it can consist of manipulatives, symbols or utterances/ words),
- 'object' or 'reference contexts' stands for aspects of knowledge explicitly or implicitly recurred to for explaining the 'sign' and
- 'concepts' – or conceptual nuances – are learners' individual notions with regard to specific mental models being necessary to interpret the 'signs' (see Steinbring 2005).

For analysing these epistemological processes, especially language means with a *cognitive function* were focussed, meaning language means necessary for thinking and verbalizing the understanding and interpretation of mathematical objects (see Prediger, Kuzu, Schüler-Meyer & Wagner 2019; Kuzu 2019) – thus, so-called 'languaging'-processes (a neologism of the words 'language' and 'thinking', expressing the closeness of both aspects) were focussed and included in the midst of the Epistemological Triangles (see Swain 2006).

3.3. Sample of the study

In the study, n = 18 students from grade 3 to 6 participated. These learners were 9 to 12 years old and partly in primary school and partly in secondary school since the broader research question of the study is to examine the pre-algebraic understanding of the ‘Auxiliary Task’ in the transition from primary to secondary school (see Kuzu 2022b). The students participated in two videographed sessions with a small group design, where one interviewer and two students were present and solved the given tasks. These videographed sessions were then transcripted according to the transcription norms being developed at the Institute of Research and Development in Mathematics Education (see Kuzu 2019): The utterances were transcripted in an utterance-based structure with the usage of square brackets […] to describe gestures, particularities etc. since such an utterance-based structure was necessary for a turn-by-turn analysis (see section 3.2.)

The small groups consisted of medium-achiever learners from the same classes, being based on the teacher-evaluation. The decision to focus medium-achievers was made because of the necessity to gain explorative insights into interpretational processes of learners (see section 3.2) and as a starting point for such insights, medium-achievers are better suited since insights into processes of high- or low-achievers are too specific and make more sense in further evaluation steps. Since the interviews took place under pandemic circumstances, the learners wore masks, had to sit with a specific distance, windows had to be opened frequently and a necessity to cleanse and disinfect enactive materials etc. was given, and these aspects have to be considered as possibly interaction-affecting factors. Furthermore, the videographed scenes were sometimes not fully understandable (although external microphones were used) because of these pandemic aspects, which is marked in the transcripts through the code “…[not understandable utterance]” in case of necessity.

In this article, the learners S1 (9 years old, primary school), S2 (9 years old, primary school), S3 (12 years old, secondary school) and S4 (12 years old, secondary school) are focussed. S1 and S2 are one small group, being analysed in section 4.2., and S3 and S4 are another small group, being analysed in section 4.1. They were chosen because of their full participation in the sessions, a good understandability of the videographed sequences and due to their active participation in the sessions, which was an important aspect for a turn-by-turn-analyzability (see section 3.2).

3.4. Research questions

In prior analyses, the forms, conditions and first pre-algebraic thinking processes in the context of the ‘Auxiliary Task’ could be reconstructed (see section 2.2), but what has yet to be examined is which further individual notions as well as ‘Grundvorstellungen’ of mathematical objects and processes might be involved when pre-algebraically generalizing the ‘Auxiliary Task’ (see Kuzu & Nührenbörger 2021; Kuzu 2022a; Kuzu 2022b). The importance of this question is directly related to the ‘cognitive gap’ since the ‘Auxiliary Task’ may serve as linking ‘bridge’ between mathematical topics such as ‘arithmetic’ and ‘algebra’ in case of a viable proceptual understanding, which is related to ‘Grundvorstellungen’ (see section 2.1). To give insights into learners’ individual notions with regard to possibly involved ‘Grundvorstellungen’, this article focusses on following successive research questions:

Q1: Which forms of pre-algebraic generalizations can be reconstructed in the learners’ individual notions of the ‘Auxiliary Task’?

Q2: Which ‘Grundvorstellungen’ are of relevance in these reconstructed pre-algebraic generalizations of the ‘Auxiliary Task’?

Q3: Which role does a pre-algebraically generalized proceptual understanding of the ‘Auxiliary Task’ and the involved ‘Grundvorstellungen’ have in the ‘cognitive gap’ between ‘arithmetic’ and ‘algebra’?

4. Results: ‚Grundvorstellungen‘ in pre-algebraic generalizations of the ‚Auxiliary Task‘

In both sequences presented in this article (section 4.1. and 4.2), the learners had the task to describe in their own words what they think when they use the ‘Auxiliary Task’ (Task a) and what the ‘rule’ behind the ‘Auxiliary Task’ is (Task b). Both tasks were given following upon a phase where the students enactively used discrete-cardinal objects to place several ‘Auxiliary Tasks’ and thus had to reflect on the proceptual meaning behind the strategy (see section 2.3).

4.1. ‘Numbers-as-indeterminate’ and the ‘compensation rule’ in both rounding directions (S3 & S4)

In this sequence, the learners S4 and S3 are talking about the ‘Auxiliary Task’ for the arithmetic of addition. Thus, rounding up the first or second number means that *one adds too much which has to be take away* from the interim result and rounding down the first or second number would mean that *one has not added enough and has to add a still missing amount* (depending on how much was taken away) to the interim result (see section 2.3). The concrete task they are talking about and which is visible on the worksheet is 6545 + 1227. After calculating this task with the 'Auxiliary Task', there was one last subtask: They had to write down what they were thinking when using the 'Auxiliary Task' (see figure 9). This is where the sequence starts.

Person	**Turn**	**Original Transcript**	**Translation**
S4	22	Man muss drauf achten, dass man nicht vergisst, das abge-, also was man aufgerundet hat, die Zahl wieder zu subtrahieren.	One has to make sure that one does not forget, that whi-, well what one rounded-up, to subtract that number again.
S3	23	Ja, ja.	Yes, yes.
I	24	Ja.	Yes.
S3	25	Also es gäbe jetzt noch so ein paar dumme Fehler, die mir einfallen, wie zum Beispiel dass, man nehme ich jetzt einfach mal 1227. Dass man aus Versehen, wenn man jetzt einfach ganz schnell durchrechnet zu 1220 rundet, aber dann noch 7 abzieht. Dann hat man auch wieder das falsche Ergebnis.	Well, there also are some stupid mistakes I could think of, for example that, let us now say 1227. That one unintentionally, when one calculates simply very fast rounds to 1220, but then takes away 7 again. Then one has the wrong result again.
I	26	Ja. Das sind dann alles so noch so kleine Rechenfehler, die sich dann auch noch nebenbei einschleichen können.	Yes. Those are then so/ like small calculation mistakes, which can slip in incidentally.
S3	27	Ja.	Yes.
[Off-topic]			
I	30	Okay, so, nochmal zurück. Habt ihr denn irgendwie eine Idee bei welchen Zahlen das auf jeden Fall besonders Sinn macht und bei welchen vielleicht eher nicht so? Oder sagt ihr das ist komplett egal?	Okay, now back again. Do you have any idea now for which numbers that makes sense particularly and for which numbers it does rather not? Or do you say that is totally irrelevant?
S3	31	Also bei Zahlen ehm. Sagen wir jetzt hier würde mal anstatt 1227 würde da 1221 stehen. *[zeigt mit dem Finger auf die Aufgabe 6545 + 1227]* Dann könnte man den Trick noch so ein bisschen abändern, dass man ehm erst was subtrahiert, dass man am Ende dann noch addiert. Dann hat man im Grunde das gleiche nur irgendwie ja umgekehrt oder sowas.	Well for numbers ehm. Let us say now that it would be 1221 instead of 1227 *[points at the task 6545 + 1227 on the work sheet]* Then one could change the trick slightly, that one at first takes away something which has to be added at the end. Then at the end, basically one has the same but somehow reversed or something like that.
I	32	Ja.	Yes.
S4	33	Das geht natürlich auch. Das brauchst du nicht, wenn da jetzt direkt schon steht 1230. Dann brauchst dus nicht.	That works too of course. You do not need that, if there would be directly 1230 now. Then you do not need it.
S3	34	Ja, dann brauchst dus nicht *[nickt]*	Yes, then you do not need it *[nods]*

At the beginning of this sequence, in turn 22, S4 starts with a first generalized expression of the rule behind the 'Auxiliary Task' by using words like 'numbers', which shows a first form of detached, non-concrete thinking of numbers (see Steinweg, Akinwunmi & Lenz 2018; Kuzu 2022b): S4 emphasizes that there is something very important, which should not be forgotten, and that is to subtract that number which "one rounded-up" (see turn 22). Here, S4 might refer to two things: Firstly, to take away the *whole* rounded-up number (e.g., 1230 if rounded up from 1227) or secondly, to take away the amount which was used for rounding up, that is "3" if one rounds 1227 up to 1230. "*What* one rounded-up" in combination with the word "again" sounds like the latter since the addition of + x (or + 3 in case of 1227) corresponds directly to the necessity to subtract – x (or – 3 in case of 1230), which could be interpreted as "*that* number again" (see turn 22). Still, it is not clear yet of what exactly S4 thought when explaining her rule. In turn 23 and 24, S3 and the interviewer agree with S4s explanation, and directly after that, in turn 25, S3 describes some 'typical mistakes' or rather one mistake: To round down from 1227 to 1220, but *"then take[s] away 7 again"* (see turn 25). Here, S3 refers to an important mistake with regard to the compensation rule: Rounding *down* would mean that one has *to add again* what still has to be added and not to take away again 7. At the same time S3 now explicitly refers to the amount or number needed to round down (+ 7) and not to the whole number (1220) and since S4 does not contradict his explanation, it seems to be viable to assume that S4 also thought of "that number" in turn 22. In turn 26 then, the interviewer reinforces S3s answer in turn 25 (and indirectly S4s explanation in turn 22) by speaking about "small calculation mistakes" (see turn 26). After that, S3 proclaims "Yes" confirmingly (see turn 27).

After a small off-topic discourse from turn 27- 30, the interviewer now asks the learners if they can think of specific numbers, where the strategy especially makes sense, rather no sense or if that is "totally irrelevant" (see turn 30). It might be intended as an open question, but it seems to be slightly tendentious since the learners might know that – after all the tasks being solved about the 'Auxiliary Task' before this question – that it cannot be irrelevant and that the Interviewier thus might expect a positive answer to the question. In turn 31 then, a positive answer is given: S3 does not give an example of a term where it does *not* make sense to use the 'Auxiliary Task' but instead gives another example where it makes sense by changing the numbers in the task ("1221 instead of 1227"). Interestingly, now S3 not only gives another example but he also changes the *direction* of compensation. He states that "one could *change* the trick slightly" by taking away something "which has to be *added* at the end" (see turn 31). Up until now, the students talked about a compensation through *taking away* the rounding number (see turn 22- 25) – thus being contextualized in a cardinal situation (an ordinal contextualization in contrast would lead to the usage of language means like 'the big jump' and 'the small jump') – which would be necessary when rounding up, but now another 'case' is mentioned where the number can be rounded down (see figure 7).

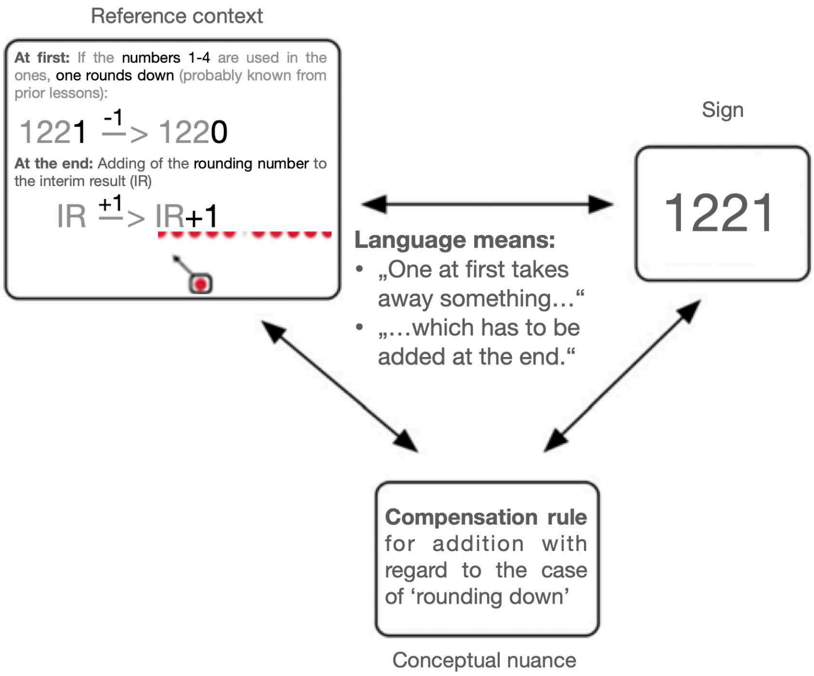


Figure 7. The epistemological triangle for S3s utterance in turn 31

In figure 7, S3 now looks at a slightly changed sign he introduces to the discourse independently – the number 1221 instead of 1227 in a cardinally situated explanation – and by probably referring to a known rounding-down rule, he describes how he has to "take away" a specific amount (see turn 31). Since S3 described a viable rounding-down process in turn 25 – thus only few turns prior – , one might presume that he again thinks of rounding down to the next tens, which would be 1220 by "taking away" 1 (see turn 31). His statement "has to be added *at the end*" (see turn 31) would then mean to calculate +1 at the end. This hypothesis is affirmed by the following part of the utterance, where he states that "basically one has *the same* but somehow reversed" since he now directly links his explanation to his prior explanation which was also about rounding down (thus 'the same'). The term "reversed" seems to be related to the reversed compensation process he also describes in the prior sentence of the same utterance ("which has to be added at the end", see turn 31). Since S3s explanation is framed in a cardinal logic, a nonconcrete amount of red dots and a concrete addition number (here +1 red dot) is also depicted in the reference context. His utterance "or something like that" seems to indicate uncertainty regarding one or more words he uses, probably with regard to the word "reversed" (see turn 31). The word "reversed" might not be fully appropriate, either because it is not a mathematical language mean (he maybe anticipates a teachers' expectation of using 'correct' or rather 'accepted' mathematical terms) or because it does not fully 'catch' his way of thinking. In fact, not everything about his explanation is *reversed*: The rounding down process, for example, is the same and only the last compensation step is reversed, which might also be a reason for his unsafety. Nevertheless, with regard to the conceptual facet of the epistemological triangle, he describes the *compensation rule for addition with regard to the case of 'rounding down'* in a viable, yet unconcrete way by using the underdetermined language mean "something which" (see turn 31).

After turn 31, it goes on with a positive affirmation through the interviewer (“yes”) as well as through S4 in the first part of turn 33, where S4 confirms the new ‘case’ S3 gives in turn 31 (“That works too of course.”) before stating another ‘case’: “if there would be *directly* 1230 now” (see turn 33). This is a special case in so far as that is already rounded down (if it follows S3s logic from turn 31) or rather rounded up (if it is referring to the number 1227), which would mean that “you do not need *it*” (see turn 33). The term “it” again is underdetermined, but it seems to refer to the process of compensation since in this special case that would be the logical aspect to become unnecessary (see figure 8).

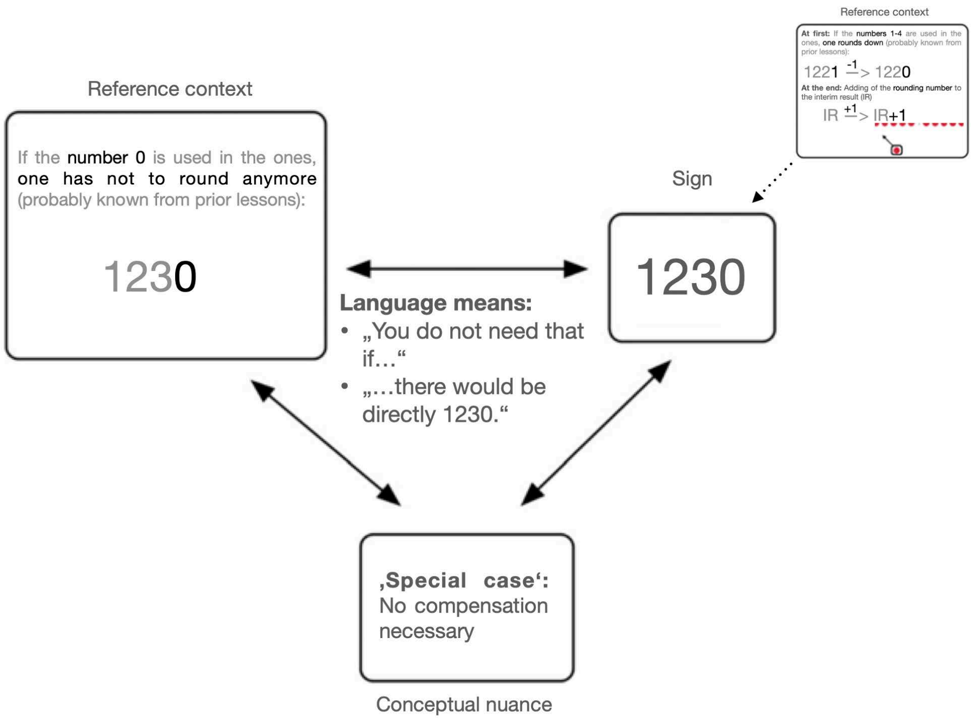
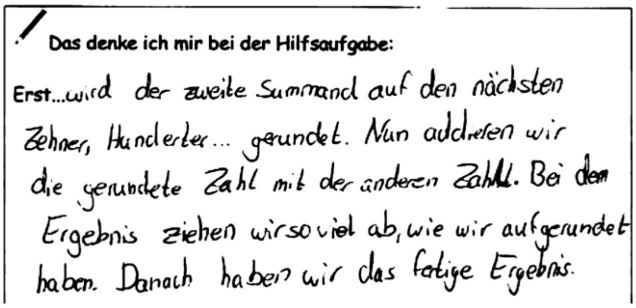


Figure 8. The epistemological triangle for S4s utterance in turn 33

In figure 8, the new sign S4 constructs is 1230 – being linked to the reference context of S3, where numbers with a “0” in the ones were constructed in the second step (see turn 31) – and as a reference context, a ‘special case’ in the context of rounding-tasks is given: the case of ‘not having to round up or down’ because of a number, which has already a “0” in the ones (for rounding processes with regard to ones). From a conceptual perspective, it might be regarded as ‘special case’ since it is a case where no compensation is necessary because of a number in a specific form: A number with the number “0” in the ones. Thus, what S4 does is to give more cases being relevant to an ‘analytical noticing’ before deciding to use the ‘Auxiliary Task’, or rather when not to use it. Right after S4’s addition of this ‘special case’, S3 confirms it with “yes, *then* you do not need it” (see turn 34) and the emphasis of the word “then” might refer to the particularity of the case (a number with a “0” in the ones).

At the end of this discourse, the learner S4 notes down a written answer in cooperation with S3 since that was the last step in the task (see figure 9).


Das denke ich mir bei der Hilfsaufgabe:
Erst...wird der zweite Summand auf den nächsten Zehner, Hunderter... gerundet. Nun addieren wir die gerundete Zahl mit der anderen Zahl. Bei dem Ergebnis ziehen wir so viel ab, wie wir aufgerundet haben. Danach haben wir das fertige Ergebnis.

Translation of the task: *“This is what I think when using the ‘Auxiliary Task’: First…”* Translation of the written answer: *“[First…] the second summand is rounded up to the next tens, hundreds... Now we add the rounded-up number to the other number. At the result we take away as much as we rounded up. Then we have the finished result.”*

Figure 9. The written answer of S4

In this written answer (see figure 9), both learners condensate the prior interaction from turn 22-34, but in a more explicit way. Firstly, they refer to “the second summand” – a typical word used like a verbal variable since it stands for a plurality of possibilities (see Steinweg, Akinwunmi & Lenz 2018) – , and then they state that they round this second summand up to the "next tens, hundreds…". Interestingly, now they include further cases they didn't talk about in turn 22-34: The rounding up of hundreds – not only of ones – and by writing the abstract sign "…", they indicate that it could also work for thousands, ten-thousands etc. Here, the "…" is a generalized expression for a plurality of possibilities where the 'same rule' works: The compensation of what was added or taken away when rounding up or down (both cases were described by the learners in the interaction from turn 22-34). The "…" sign is even more abstract than the verbal variable "the second summand". After that, the students refer to "the rounded-up number" and describe that this number is to be added to the "other number". Again, the language means "rounded-up number" and "other number" are verbal forms of non-alphanumerical variables, they stand for a plurality of possible objects. In the last part of the written answer, they seem to refer to the interim result by using the language mean "result" – which is slightly ambiguous since it also could refer to the end result – and state that one has to "take away as much as" one rounded up to get to the "finished result", being analogous to the utterance in turn 22, where the process of subtracting the rounding number is described, and turn 25, where the language mean "taking away" is used explicitly.

Looking at the meaning-related language means used in this written answer as well as in the interaction from turn 22-34, it is noticable that a lot of terms and utterances standing for a plurality of possibilities are used:

- In turn 22, where S4 describes that "what one rounded-up, to subtract that number again": This could be related to rounding-up processes with a 5, 6, 7, 8 or 9 in the ones and then the same 'rule' would apply ("to subtract that number again").
- In turn 31, where S3 mentions "for numbers" and generates an own example after that for the 'case' of rounding down, by stating "that one at first takes away something which has to be added at the end": This general explanation would also work for the numbers (1,) 2, 3 and 4 since in all cases, one has to round down and compensate by adding what was rounded down at the end (which S3 also highlights).
- In the written answer, where they use variable-like words when writing about "the second number", "the rounded-up number", "the other number" etc.: All of these language means are non-concrete expressions of numbers. S3 and S4 do not mention one concrete number in their explanation, but it is still a viable explanation without concrete cases, thus they describe a generalized 'rule' working for a plurality of numbers.

In all of these cases, the learners S3 and S4 treat numbers as interchangeable objects with regard to the rule they describe. They use the verbal means referring to numbers or parameters like *numbers-as-indeterminate*, meaning that the parameters they describe by using terms like "second number", the "rounded-up number" etc. could stand for a plurality of numbers. It is a pre-algebraic 'Grundvorstellung' resulting from the generalized, proceptual understanding of the arithmetical operations behind the 'Auxiliary Task', which is analogous to the algebraic 'Grundvorstellung' of variables-as-indeterminate (see Akgün & Özdemir 2006; Korntreff & Prediger 2022), where alphanumerical symbols do symbolize the plurality of possibilities, but in this sequence, S3 and S4 manage the same with their non-alphanumerical language means (aside from the abstract "…" sign). All three facets of 'Grundvorstellungen' can be reconstructed here: First, an adequate mental representation of the compensation process is visible when S3 and S4 describe the rounding up and rounding down process followed by a viable compensation (see turn 22 and 31), second, a mentally represented prototypical activity is reconstructable in the different steps of the 'Auxiliary Task' – from the modification in the first step to the compensation in the last step (see written answer of both students and turn 31) – and third, a contextualized thinking is visible when the students speak of "taking away" objects, which is situated in a cardinal and everyday thinking of the subtraction (see turn 25). Furthermore, a specific form of 'indeterminacy' emerges here since the language means representing numbers do not stand for specific numbers but a generalized view on numbers – for a plurality of possible numbers – , which is a key component of pre-algebraic thinking (see Radford 2018) and this sequence shows that, even if students do not use alphanumerical symbols in primary school, they can a) express an indeterminacy of numbers and b) explain the 'compensation rule' by doing so (see Kuzu 2022a; Kuzu 2022b). Thus, even though it is only a sequence about the compensation rule for the arithmetic addition – the 'rule' has to be reflected in the context of other arithmetical operations in further steps – , an important 'bridge' is built here between the topics 'arithmetic' and 'algebra' through the activation of the 'Grundvorstellung' of *numbers-as-indeterminate* and the *proceptual reflection of the compensation rule*.

4.2. 'Numbers and terms as highly manipulable objects' and the 'equivalence of terms' (S1 & S2)

In this section, S1 and S2 had the task to formulate a 'rule' for calculating with the 'Auxiliary Task', it is the same task as in section 4.1. They start to discuss the 'rule' immediately after the interviewer reads the task from the task sheet. The task they solved before was 35 – 18 (it is still visible on the table). Their task had smaller numbers than the task of S3 & S4 because of the age difference and the fact that they were in primary school.

Person	Turn	Original Transcript	Translation
S2	11	Also das wäre	Well that would be
S1	12	Ehm achtzehn, kann ich da die Zahlen *[zeigt auf das Arbeitsblatt]* achtzehn plus zwei sind gleich zwanzig minus zwei dann schreiben?	Well eighteen, can I write there *[points at the task sheet]* eighteen plus two makes twenty minus two then?
S2	13	Also ich hätte gesagt, die Regel ist, ja zum Beispiel, ja du darfst dir jetzt noch zwei dazunehmen *[wischt mit mit ausgestreck-tem Zeige- und Mittelfinger der rechten Hand nach rechts]* zu der Aufgabe, dann musst du aber *[hebt den rechten Zeigefinger nach oben]* auch wieder zwei dahin machen *[wischt die rechten gestreckten Zeige- und Mittelfinger wieder nach links zurück]*. Das wäre dann vielleicht die Regel *[hebt beide Hände geöffnet nach oben]*?	Well, I would have said that the rule is, well for example, yes, you may take two extra *[swipes right with her strechted out index and middle fingers of the right hand]* to the task, but then you must *[takes up the right index finger]* also put two there again *[swipes back left with her stretched out index and middle fingers of the right hand]*. That would then be the rule probably *[opens and lifts her both hands up]*.
I	14	Mhm. Das hast du jetzt gut erklärt. Ihr gemeinsam habt das gut erklärt, ne.	Mhm. You have explained that well now. You both have explained that well, right.
S1	15	*[nickt]*	*[nods]*
S2	16	Mhm *[nickt]*.	Mhm *[nods]*.
I	17	Jetzt habt ihr gesagt, was ihr euch gedacht habt. Ich glaube das müssen wir auch gar nicht mehr aufschreiben, weil das haben wir jetzt auch schon so viel erklärt, ne.	Now you have said what you thought of. I think we do not have to write that down now, because we explained so much now, right.
S1	18	*[nickt]*	*[nods]*
I	19	Es reicht, wenn ihr das jetzt einmal gesagt habt. Ehm aber vielleicht ehm kann, können wir das ja, was du gesagt hast *[zeigt auf S2]*, nochmal aufschreiben. Also du hast gesagt, erst, also man kann sich eine Zahl dazu denken, muss die aber, wenn man sie zu viel abgezogen hat, wieder	It is sufficient if you have said it once now. But ehm maybe ehm you can, we can, what you said *[points at S2]*, write that down. Well, you have said, first, well, one might think of an extra number/ of a number in addition, but then you must, when you have taken it away too much, again
S2	20	zu der anderen Zahl *[wischt ihre beiden Hände nach links]*, sage ich jetzt mal.	to the other number *[swipes both of her hands to left]*, that is what I'd say.
S1	21	Also plus wieder.	Well, plus again.
I	22	Okay.	Okay.
S2	23	Ich weiß nicht genau, vielleicht gibt es dafür einen Begriff, Nachbarzahl oder so *[hebt beide Hände geöffnet nach oben]*. Ich sage jetzt mal zu der Zahl *[wischt mit den beiden Händen nach links]*, die- zu der anderen Zahl *[legt die beiden Hände links von ihr auf den Tisch]*, anderen Zahl.	I don't know, maybe there is a word for that, neighbour-number or so *[opens and lifts her both hands up]*. Well, I'll say now to the number *[swipes with both hands to left]*, the- to the other number *[lays her hands down on the table left from her]*, other number.
S1	24	Ist die nächste, nächstliegende Nachbar-zehner-Zahl.	

In turn 11, S2 seems to start to think about "what would be" and since that happens after the interviewer reads the task, one could assume that she now thinks about a possible 'rule'. In turn 12, S1 then starts with a concrete example: She asks the interviewer if she may take the task eighteen plus two, or rather if she can write it down. The reason for this question may be an uncertainty if she can directly write down the solution since the task was to talk about it first and then to write it down. Moreover, the task eighteen plus two does not match the form of the 'Auxiliary Task' but rather the form of the second number and the rounding-up process, which S1 might have thought of here. The fact that the number eighteen matches the second number in the term 35 – 18 (lying visibly on the table) supports this assumption, S1 seems to be focussing the second number.

S2 then starts to give an example in turn 13, but not by asking if she can write it down, but by stating that she "would have said" what the rule is (which is in direct opposition to S1's question if it would be okay to write it down). Interestingly, she does not only oppose the question of S1 but also changes the 'direction' in formulating the rule: By stating that one could "for example […] take two extra", she gives a partly concrete and partly generalized example. She gives a concrete example for the rounding-up number ("+2"), but deconcretizes the second number with the language mean "to the task", which is not the concrete number "18" anymore, as S1 suggested in turn 12, but could be *any* number. It could also be referring to the second number or the first number, so the rounding-up process is regarded in a detached way, although she uses the same 'case' of adding two to a number (which might also stand in direct relation to the task solved before, 35 – 18, since the students had to add two extra there too). She then explicates the last compensation step by stating that one has to "put two there again" (see turn 13), which at first seems to be underdetermined, but a possible indication of what she might mean by "put two there again" lies within another 'channel' of transporting meaning: It is noticeable that she uses a lot of possibly meaning-related gestures. Gestures are an important 'mode' of representing mathematical meaning and thinking and when students accompany their verbal explanations with gestures, this should be analysed in detail as well (see Robutti, Sabena, Krause, Soldano & Arzarello 2022). Analysing S2s usage of gestures, two fields of vision become reconstructable: When speaking of "two extra", S2 swipes right with her index and middle finger, when mentioning to "put two there again", she swipes left with her index and middle fingers. These 'directions' of thinking the rounding-up and compensation process seems to be linked to the way she used discrete manipulatives in the proceptual tasks before being asked to formulate the 'rule' (see figure 10)

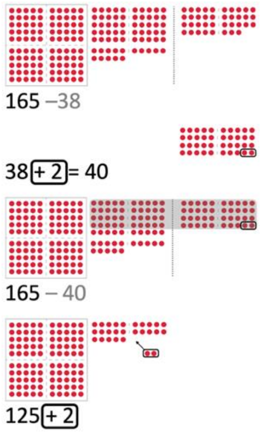


Figure 10. The discrete-cardinal representation from the task sheet (which was used enactively with wooden objects before the depiction on the task sheet).

In figure 10, it is clearly visible that on the *right side*, the second number is depicted in a discrete-cardinal way and on the *left side*, the first number is depicted in a discrete-cardinal way (see section 2.3). Furthermore, after taking away the second number from the first number, the interim result is also visible on the *left side*. Thus, the gestures might give a possible explanation for the underdetermined language means "take two extra to the task" and "two there again": It could be that she thinks in relation to the manipulatives and iconic representations from the prior tasks, meaning that she mentally may think of the second number on the right side (for adding "two extra" as she says in turn 13) and of the interim result on the left side, where one has to "put two there again" for a viable compensation since two were taken away too much after rounding up (see turn 13). Another important aspect in turn 13 is the specific way S2 speaks of the process of utilizing the 'Auxiliary Task': She states that one "*may* take two extra" – meaning that one could also not take two extra, thus she describes a voluntary or optional act – but then, if you decide to take two extra, you "*must* also put two there again", which is not a voluntary act anymore, but a necessity being represented through the language mean "must" (see turn 13).

In turn 14 then, the interviewer reinforces S2's more general explanations at first, which at the same time would mean – consciously or unconsciously – an ignorance of S1s question in turn 12. Probably realizing this possible conflict, the interviewer then switches to a collective reinforcement by stating that *both* of them "have explained that well" (see turn 14). After that, S1 and S2 seem to be pleased since both nod positively and in turn 17, the interviewer goes on by stating that they "do not have to write that down now" since they explained so much, which is another reinforcement as well as a 'reward' in form of a skipped task: Normally, the students would have to write down the answer, which again lead to a nodding by S1 in turn 18. The interviewer then contradicts himself in turn 19 by repeating at first that a verbal answer is sufficent and then stating the opposite: that the learners could write it down anyway (the interviewer probably might be unsure if 'skipping' the writing down task is a good idea). He then rephrases what the students said until then by saying that "one might think of an *extra number*/ of a number *in addition*", which one then would have to to add to the interim result. The last aspect – the adding of the compensation value to the interim result – is not pre-empted by the interviewer but rather left open as a typical 'filling out' spot in the utterance in turn 19 since he stops and looks up to the learners. Understanding this 'signal', S2 then completes the interviewer's utterance in turn 20 by referring to the "other number" and simultaneously swiping left, which again would be the side where the interim result is visible on the task sheet (see figure 10). This again reinforces the hypothesis that she might think analogously to the two sides of the discrete-cardinal icons on the task sheet.

The interaction then goes on and S1 seems to correct, or rather reformulate S2's implicitly thought of operation in the utterance "to the other number" by stating "plus again" (see turn 21), which the interviewer confirms with an "okay" in turn 22. In the next turn then, in turn 23, S2 again uses a similiar explanation to her two previous explanations by again referring to the her field of vision, but this time, she only emphasizes the left side: She states that now something has to be done "to the number" on the left side (by swiping left) and repeating that in a second sentence immediately after that by stating "to the other number" (by again swiping left). Since her explanation follows upon the interviewers 'filling gap' in turn 19, where the compensation process is focussed, she might focus the compensation also, which would explain her swiping left since there would be the interim result where the compensation has to be done (see figure 11).

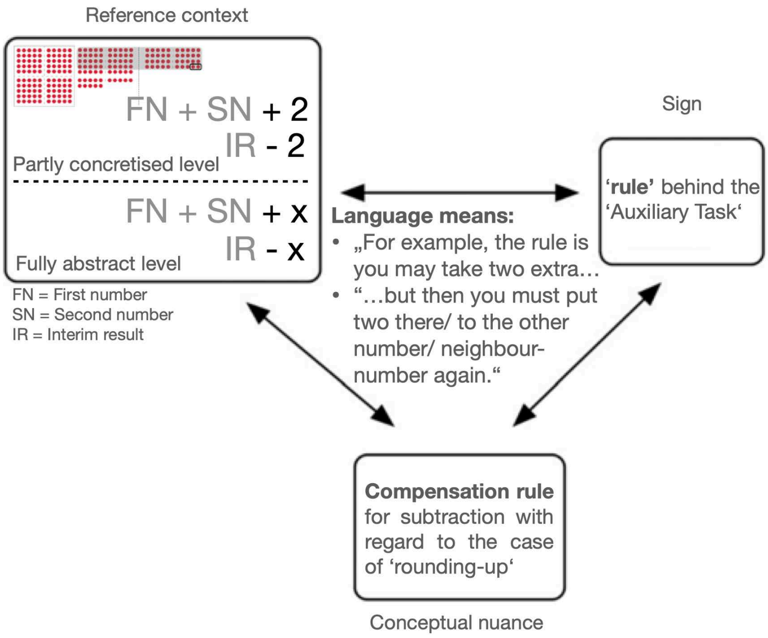


Figure 11. The epistemological triangle for S2's utterance in turn 23 (see also turn 13 and 20)

In figure 11, an epistemological triangle representing S2's explanation from turn 13, 20 and 23 is constructed. Since S2's utterance in turn 23 is linked to aspects from her utterance in turn 13 and 20, or rather continues them, the epistemological triangle summerizes these three turns and thus includes aspect from all turns. The sign which is leading to S2's explanation is the language mean 'rule', which is asked by the task (and the interviewer). It is a specific form of signs, a language mean standing for an explanation of the mathematical structure behind a concrete process, which is a highly complex process from a linguistic and logical viewpoint (see Clarkson 2004; Hein 2019). S2 interprets this sign by using specific language means standing for generalized numbers – like "the other number" (turn 20) or "neighbour-number" (turn 23) – as well as concrete numbers like "two extra" (turn 13). She is referring to what one may do when using the 'Auxiliary Task', thus to the reference context of the the process of modifying the second number "on the right" (see turn 13) and compensating it at the interim result "on the left" (see turn 20 and 23). Since she gives plus two *as an example* for a possible modification (see turn 13), it is assumable that she might have a first structural understanding and would apply the same 'rule' to other examples like +3, +4 etc., which is why on an abstract level, + x is part of the reference context. Being only assumable at this point, later analyses show that she can in fact give further examples: In later sequences, she gives another example to the 'rule', but this time with + 8 for modifying the second number and – 8 at the interim result (not visible here in this interaction sequence). Her explanation in turn 13, 20 and 23 is thus *partly concretised* due to the fact that she exemplifies + 2 as a possible modification value on a generalized number (on the "number on the right", see turn 13) and partly more abstract because of the usage of language means for representing a plurality of numbers (which is again an interpretation of numbers-as-indeterminate, being similiar to S3's and S4's language usage in section 4.1.). What S2 does is to give a so-called generic example: It is a carefully chosen, representative example or object for explaining the 'rule' and this explanation would work for similar objects (see Lew, Weber & Mejía Ramos 2020). The conceptual meaning being explained here is the compensation rule for subtraction with regard to the case of 'rounding up' the second number.

Although S2 seems to use a combination of language means and gestures again in turn 23, her explanation is still highly underdetermined: It seems probable that she refers to the way the discrete-cardinal material was used, but that cannot be confirmed definitively in this sequence. Again, further analyses of the progession of S2 confirm this hypothesis (but it cannot be evaluated finally in this sequence). In the last turn, in turn 24, S1 again tries to correct S2 by using the inventend word “neighbour-tens-number”. S1s ‘grasping’ for the right words as well as S2s attempts to offer these – although his words are also invented (see turn 24) – shows how hard it is for learners to verbalize the ‘rule’ behind the ‘Auxiliary Task’, which is a finding prior analyses also confirm (see Kuzu & Nührenbörger 2021; see section 2.2).

To summarize the turn-by-turn analysis in section 4.2., the learners S1 and S2 show a thinking of ‘numbers and terms as highly manipulable objects’, which is slightly differing from the *numbers-as-indeterminate* ‘Grundvorstellung’ (see section 4.1.) in so far as that not a plurality of possible numbers for a specific parameter like the first number or the second number is focussed, but a plurality of possible modifications (with consequences), e.g. an adding of numbers extra as S2 describes in turn 13, 20 and 23. Such a view emphasizes that you *can* modify numbers always by adding numbers extra, but then you must consider the consequence, that is to compensate what was added by taking it away when utilizing the ‘Auxiliary Task’ for subtraction. It is a *flexibilized* view on numbers going beyond an analytical noticing with a focus on rounding up or down processes – you can leave numbers as they are or modify them, both is possible – , which is also linked to all three facets of a ‘Grundvorstellung’: To a viable notion of compensation (see figure 11), a mentally represented prototypical activity of modifying and compensating (see turn 20 and 23) and to a contextualized thinking in form of a cardinally situated usage of language means, e.g. “taking extra” (see turn 13). That is an important difference to a numbers-as-indeterminate ‘Grundvorstellung’, but also highly relevant to pre-algebraic thinking processes because it may lead to other forms of the ‘Auxiliary Task’, where not a rounding up or rounding down process is focussed but *every* modification, e.g., the complement building (see Kuzu 2022b). At the same time, S1 and S2 show a first ‘Grundvorstellung’ of the ’equivalence of terms’ in this sequence, which is also visible turn 13, 20 and 23, but with regard to the gestures used by S2: She indicates a two-sided operation in her field of vision and it seems that she thinks of the second number, which can be modified, on the *right side* of her vision, and of the interim result, where the compensation has to be conducted, on the *left side* of her vision, and this matches the way the manipulatives and objects where used in the learning environment. This is another important pre-algebraic facet since it is preparatory with regard to a fully algebraic introduction to equations (see Schwarzkopf, Nührenbörger & Mayer 2018): Learners, who might develop an important sense or a viable understanding of the necessity to compensate modifications because of a notion of equivalence of terms can build upon their viable prior experiences in later grades. This is not a singular insight: Other learners showed a similar equation-like thinking when describing a necessary equivalence between the first term and the compensated term (see Kuzu 2022b).

5. Discussion

With regard to the research questions Q1- Q3, specific insights could be given in the analyses in section 4.1 and 4.2., which add up to and deepen prior results from this study (see section 2.2).

Concerning Q1 – an analysis of the forms of pre-algebraic generalizations – all learners (S1- S4) tended to explain the ‘rule’ behind the ‘Auxiliary Task’ by using variable-like language means – for example by using language means like “rounded-up number” (see section 4.1., turn 22), “second summand” (see section 4.1, written answer), “the other number” (see section 4.2, turn 20) and “neighbour-number” (see section 4.2., turn 23) – , and by using these language means, they showed a (first) generalized view of numbers: They described the ‘rule’ behind the ’Auxiliary Task’ as a process being applicable to a plurality of numbers and terms. Furthermore, especially S2 showed a gesture-based and equation-like thinking of two sides of a termic modification and compensation when explaining the ‘rule’ behind the ‘Auxiliary Task’ (see section 4.2), which is another important pre-algebraic facet: The occurring inequivalence after modifying one number has to be kept in mind for the compensation step and if learners do forget it – which occurs frequently when introducing into the strategy – , the compensation rule is not used in a viable way. Not only a compensation, but also the right compensation, meaning a compensation in the ‘right direction’ is important and this was an aspect not only S2 showed in her explanation in turn 13, 20 and 23, but also S3 when giving an example for an ‘Auxiliary Task’ for rounding down the second number in an addition term, which changes the compensation direction: Instead of *taking away* the rounding number, it has to be *added*, which S3 identifies in a viable way (see section 4.1).

With regard to Q2 – the analysis of the ‘Grundvorstellungen’ possibly being involved in the individual notions of the learners – the emergence of pre-algebraic generalizations seems to lead to specific ‘Grundvorstellungen’: The learners developed an understanding of

- *Numbers-as-unknown:* They found out missing numbers like the missing rounding number in the thinking bubble, which was not visible in section 4.1 and 4.2., but in prior sequences as well as in written solutions (see figure 2).

- *Numbers-as-indeterminate:* In section 4.1. as well as in section 4.2., the learners treated the numbers as changeable parameters by using non-alphanumerical variables in form of abstract language means like “the rounded-up number”, “second summand”, “the other number” etc. (see section 4.1., 4.2. and overview regarding Q1 in this section)
- *Numbers and terms as manipulable objects:* Especially S2 described a specific form of a flexibilized view on numbers and terms by describing how one can change numbers – if one wants – but that would mean that one *must* compensate. This means that *every* number is modifiable, if adequately compensated – a highly abstract and pre-algebraic view on numbers and possible ‘modifications’ of numbers (see section 4.2)
- *Equivalence of terms:* In sequence 4.1. and 4.2., the learners described a process of compensation due to an occuring inequivalence. Most explicitly, again S2 described this process by describing the modification and compensation process verbally and through gestures by linking it to her field of vision with a ‘right side’, where the modification is located, and a ‘left side’, where the compensation is located. This dual-sided view resembles an equation-like perception of a change on one side, which has to be ‘transferred’ to the other side to maintain the equality (see section 4.2.)

These four aspects could be reconstructed in section 4.1 and/ or in section 4.2., but further analyses in Kuzu & Nührenbörger (2021), Kuzu (2022a) and Kuzu (2022b) do confirm these outcomes.

With regard to Q3 – an assessment of the role a proceptual understanding of the ‘Auxiliary Task’ has in the ‘cognitive gap’ between ‘arithmetic’ to ‘algebra’ – , thus important insights could be given through the analysis and answering of Q1 as well as Q2: A proceptually reflected understanding of the ‘Auxiliary Task’ may lead to pre-algebraic interpretations of the mathematical structure behind the ‘Auxiliary Task’, which is the compensation strategy in its different proceptual forms (see section 2.3), and to specific pre-algebraic ‘Grundvorstellungen’. To summarize the insights, the complexity of a proceptual understanding of the ‘Auxiliary Task’ comprises of

- the chronology of the steps (either simultaneous or asynchronous steps),
- important cognitive language means for articulating and thinking the modification and compensation process,
- the usage of manipulatives and representations for explaining the proceptual meaning behind the strategy – ranging from different forms of cardinal representations to ordinal representations – ,
- the focussing of different modification processes (rounding-up, rounding-down, complement building or the usage of neighbour tasks) and
- a viable understanding of the compensation process as well as the emergence of specific pre-algebraic ‘Grundvorstellungen’ (numbers-as-unknown, numbers-as-indeterminate, numbers and terms as manipulable objects as well as the equivalence of terms).

All of these facets are interrelated and should be considered, if a learning environment is designed with the goal of fostering a proceptual understanding of the ‘Auxiliary Task’ (see figure 12)

In figure 12, a complex network is visible with regard to the proceptual understanding of the ‘Auxiliary Task’. Without any claim to be exhaustive – further insights into learners proceptual interpretation of the ‘Auxiliary Task’ are still needed – , but being based on empirical insights in the analyses of this article as well as prior analyses (see section 2.2), figure 12 illustrates the new insights in the context of this article and study. The only missing involved pre-algebraic ‘Grundvorstellung’ is numbers-as-unknown – as stated above – , but that is an aspect being visible in all cases where students had to find out missing or ‘hidden’ numbers, like the rounding-up number in the thinking bubbles (see figure 2). Although the sequences in this article do not give insights into this ‘Grundvorstellung’, which is normal since the learners are asked to formulate a ‘rule’, prior sequences do confirm the frequent occurrence of a numbers-as-unknown ‘Grundvorstellung’.

What is still missing is a direct comparison of primary school and secondary school learners proceptual interpretation of the ‘Auxiliary Task’. Their verbalization and thinking might differ with regard to the usage of language means, for example more or other abstract language means could be used (as indicated in section 4.1). Furthermore, insights are needed into transfer processes: It should be analysed how learners recognise similarities between specific terms they identify as solvable by a utilization of the ‘Auxiliary Task’. Limitations arise with regard to the sample of the study: More learners’ interpretations and generalizations should be analysed – not only in Germany, but in different countries – to allow a more detailed analysis of the processes, for example in a cross-country comparison. Still, the analyses show the necessity to think of mental calculation strategies – especially of highly complex strategies such as the ‘Auxiliary Task’ – not only as ‘calculation tricks’, but to foster the conceptual understanding of the strategies.

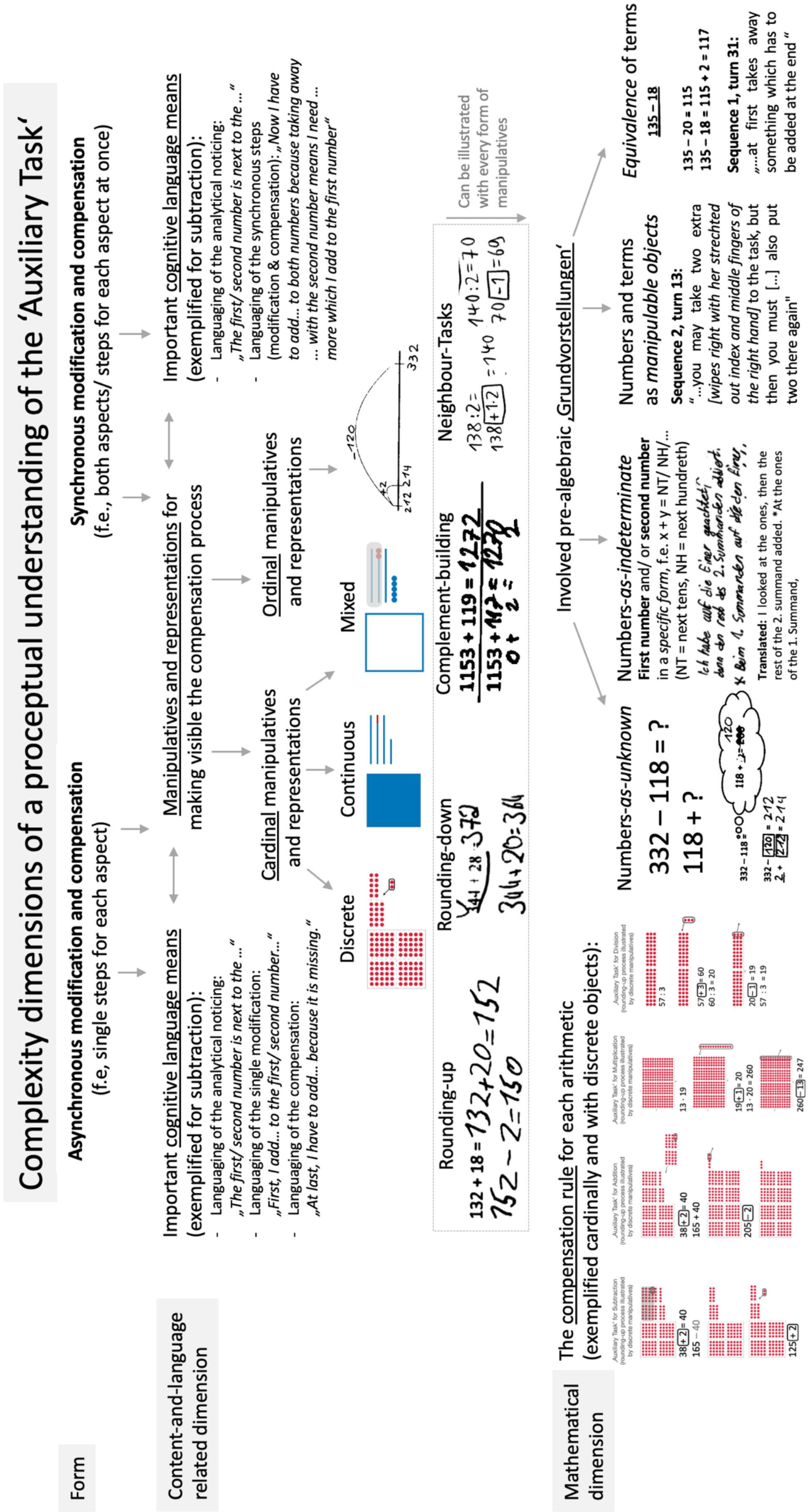


Figure 12. Complexity dimensions of the 'Auxiliary Task' (examples from the data corpus of this study)

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