Integrating Puzzles and General Problem-solving Techniques into Existing Undergraduate Mathematics Classes

Benjamin Peet

Department of Mathematics, St. Martin's University, United States (ORCID: 0000-0001-8478-185X)

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Abstract: The goal of this paper is to offer tools and tips for integrating puzzles and general problem-solving techniques into existing undergraduate mathematics classes - primarily prerequisite, lower-level classes. I discuss the research into the benefits of puzzles in mathematics education at the undergraduate level. Next, I delve into some general problem-solving techniques that offer a framework to solving puzzles, as well as provide an example set of puzzles with reference to the stated solving techniques. Finally, I set out a structure for integration of puzzles into undergraduate classes with examples for how these problem-solving techniques can be applied to traditional mathematics curriculum.

Keywords: Puzzles, Problem-solving, Undergraduate mathematics

1. Introduction

If asked to describe mathematics as an academic subject, I would use words like creative, playful, imaginative, and game-like; however, as a freshman or sophomore undergraduate student, I would have used words like functional, practical, or methodical. What changed in my perspective and why did my undergraduate self not see mathematics the way I see it now?

I propose mathematical puzzles as a possible solution to this discrepancy of feeling, and I frame the use of puzzles in the context of general problem-solving. I believe that puzzle integration into undergraduate courses can serve two interests; the appreciation of the more game-like qualities of mathematics as well as the ability to enhance existing courses by explicitly developing more general problem-solving skills, specifically student's metacognitive abilities.

In this paper, I focus on integrating puzzles into existing courses, in particular lower-level classes such as prerequisite algebra classes or precalculus courses. I aim to articulate the techniques that I have found most effective from my experiences teaching undergraduate courses. Everything touched on in this paper is intended to complement existing teaching techniques. For example, I assume active-learning techniques throughout my examples due to the substantial body of literature that establishes it as effective (Braun et al., 2017). Puzzles are naturally an active tool, and I encourage the use of active-learning techniques in their incorporation.

It is fairly common that a lower-division undergraduate math course will list something along the lines of "Students will develop new and enhance existing problem-solving skills" as a course objective. It is rare, however, that problem-solving in and of itself is taught within such a class. I think this disconnect happens for various reasons but primarily because it is difficult to assess problem-solving skills except in the context of much more specific mathematical situations. I, therefore, present my argument for integrating general problem-solving techniques within an existing mathematics class through puzzles. From there, students can then apply these problem-solving skills to more particular instances within a traditional mathematics curriculum.

I should note that much (if not all) of this paper could also apply to secondary education, but I restrict focus to undergraduate education given my experience teaching at a university. Specifically in this paper, I consider the work of Duch et al. (2001) and Pólya (1962) to outline the benefits of problem-solving education and articulate my experience in the case of undergraduate mathematics. I then consider Pólya (2004) to give a guide to specific problem-solving techniques that I adapt and adjust to consider problem-solving from a nonlinear perspective - that is, focusing on the more playful and game-like features.

I consult the work of Meyer III et al. (2014) to guide how to integrate puzzles into a mathematics course and then offer a set of 5 puzzles concerning the stated techniques. These techniques are linked to specific mathematical concepts/problems, that from my experience, they complement well. Finally, I discuss a method of integration of puzzles into courses that I have used in all of my precalculus and college algebra classes during the past four years.

2. Why puzzles? Why integrate?

If my two goals are to give students a better understanding of mathematics as playful and to improve student's problem-solving skills, the question "Why puzzles?" may be asked. Why not teach students problem-

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solving skills individually and try to give them an understanding of the game-like qualities of mathematics in a different manner?

I consult some sources to justify my use of puzzles to satisfy both goals:

In Parker (1955), Parker makes the same contention that I do. Precisely that:

> *...puzzles can be made to serve dual purposes:*
>
> *1. To secure the interest and attention of the group.*
>
> *2. To teach mathematics by illustrating and clarifying certain mathematical concepts and techniques, by securing a higher mastery of subject matter, by developing skill in manipulation, by making mathematical learnings more permanent, and by developing an appreciation of the systematic approach of algebraic methods.*

The authors in Kubinova et al. (1998) similarly state that:

> *From our own teaching experience, by using projects and mathematical puzzles I have found that our students have gained the necessary understanding, enjoyed their work and developed other important attributes such as the ability to conjecture, to work systematically and to communicate.*

As noted in Nørgård et al. (2017), enhancing the playful aspect of learning can have very strong benefits outside of mathematics. I quote:

> *...a more playful approach to teaching and learning in higher education ...stimulates intrinsic motivation and educational drive, creates safe spaces for academic experimentation and exploration, and promotes reflective risk-taking, ideation, and participation in education.*

The question of whether to integrate into existing sources as opposed to having a separate course dedicated to puzzles and problem-solving is mostly a practical one. It is certainly the easier route to integrate into existing courses than to create new ones, and the goal is to enhance the existing math courses not replace them. I contend that much of the material in a college algebra course can be made more puzzle-like and have practical uses.

For example, the notion of exponential growth can be introduced in the form of a puzzle.

Puzzle 4, later on, is such a puzzle that can be used to introduce exponential growth.

The literature also bolsters this assertion. In particular, Lester Jr (2013), gives the following principles for teaching problem-solving:

> *1. The prolonged engagement principle. In order for students to improve their ability to solve mathematics problems, they must engage in work on problematic tasks on a regular basis, over a prolonged period of time.*
>
> *2. The task variety principle. Students will improve as problem solvers only if they are given opportunities to solve a variety of types of problematic tasks.*
>
> *3. The complexity principle. There is a dynamic interaction between mathematical concepts and the processes (including metacognitive ones) used to solve problems involving those concepts. That is, heuristics, skills, control processes, and awareness of one's own thinking develop concurrently with the development of an understanding of mathematical concepts. (This principle tells us that problem-solving ability is best developed when it takes place in the context of learning important mathematics concepts.)*
>
> *4. The systematic organization principle. Problem-solving instruction, metacognition instruction in particular, is likely to be most effective when it is provided in a systematically organized manner under the direction of the teacher.*
>
> *5. The multiple roles for the teacher principle. Problem-solving instruction that emphasizes the development of metacognitive skills should involve the teacher in three different, but related, roles: (a) as an external monitor, (b) as a Lester facilitator of students' metacognitive awareness, and (c) as a model of a metacognitively-adept problem solver.*
>
> *6. The group interaction principle. The standard arrangement for classroom instructional activities is for students to work in small groups (usually groups of three or four). Small group work is especially appropriate for activities involving new content (e.g., new mathematics topics, new problem-solving strategies) or when the focus of the activity is on the process of solving problems (e.g., planning, decision making, assessing progress) or exploring mathematical ideas.*

> *7. The assessment principle. The teacher's instructional plan should include attention to how students' performance is to be assessed. In order for students to become convinced of the importance of the sort of behaviors that a good problem-solving program promotes, it is necessary to use assessment techniques that reward such behaviors.*

I find that the complexity principle makes a very strong case for integration as opposed to a stand alone course.

I also believe that developing metacognitive skills is crucial to being able to effectively problem-solve. Furthermore, this belief appears as part of three of the seven principles.

I utilize each of these principles when setting out integration into existing courses.

3. What is a puzzle? What is problem-solving?

I ask two obvious precursors to any subsequent discussion: What is a puzzle? and What is problem-solving? Any discussion of problem-solving would be entirely remiss were it not to reference George Pólya. Pólya set forth a much more structured notion of problem-solving through heuristics. It is Pólya that I to look for some definitions. In Pólya (1962), he states:

> *...to have a problem means: to search consciously for some action appropriate to attain a clearly conceived, but not immediately obtainable, aim.*

However, to this author's mind, this definition makes problem-solving appear too linear - a beginning ("clearly conceived"), a middle ("search consciously"), and an end ("not immediately obtainable, aim").

From there, I formulate my own simple definition, a *puzzle* is some unknowns along with some conditions on those unknowns. *problem-solving* is trying to discover the unknowns.

Keep in mind that any equation is certainly a puzzle, and I say "trying to find" as opposed to finding; I believe that students should be discouraged from the all or nothing approach. In my experience of mathematics research, the problems attempted are rarely "clearly conceived", and the solutions obtained are usually only partial. I would wager that such a situation is true in most disciplines - solutions are not so clear cut. In many cases there may be no solution to a problem, and finding out why is certainly of value in and of itself.

3.1. The benefits of puzzles and problem-solving education within a mathematics course

I consider what teaching problem-solving can offer to an undergraduate mathematics course. To begin this discussion, I think about what students' perspectives are.

It does not take much argument to posit that puzzles and problem-solving can be a very positive change of pace for a mathematics course. Puzzles feel fun and playful, and as I will set out later in this paper, should be low stakes. Puzzles and problem-solving therefore feel like a breather from the pure mathematical part of the curriculum.

More than simply being entertained, students anecdotally report positive impacts of puzzles on their mathematics education and general education, too. In particular, students noted that puzzles and problem-solving helped give them a more positive view of mathematics. Regardless of any improvement according to some given measure, students feel better about mathematics due to this approach. I surveyed previous students from my classes over the last eight years during which I utilized puzzles. From these questions, 74% of students responded that working with puzzles was either very or extremely useful to their mathematics education, and 63% of students responded that puzzles were extremely or very useful to their general education.

From there, I consider what the literature tells us about the benefits of teaching problem-solving techniques. In their book Duch et al. (2001), the authors propose that taking a problem-based approach to teaching is the appropriate response to the modern world. I quote:

> *In a traditional undergraduate classroom, lectures are usually content driven, emphasizing abstract concepts over concrete examples and applications. Assessment techniques focus on recall of information and facts, and rarely challenge students to perform at higher cognitive levels of understanding. This didactic instruction reinforces in students a naïve view of learning in which the teacher is responsible for delivering content and the students are the passive receivers of knowledge.*

In Liljedahl et al. (2016)- an excellent summary of research into problem-solving in mathematics education - the authors echo my sentiments:

> *Mathematics has often been characterized as the most precise of all sciences. Lost in such a misconception is the fact that mathematics often has its roots in the fires of creativity, being born of the extra-logical processes of illumination and intuition. Problem-solving heuristics that are based solely*

on the processes of logical and deductive reasoning distort the true nature of problem-solving. Certainly, there are problems in which logical deductive reasoning is sufficient for finding a solution. But these are not true problems. True problems need the extra-logical processes of creativity, insight, and illumination, in order to produce solutions.

Lastly, I need to note that problem-solving skills are what employers are looking for.

I quote from Associates (2015):

Nearly all employers (91 percent) agree that for career success, "a candidate's demonstrated capacity to think critically, communicate clearly, and solve complex problems is more important than his or her undergraduate major."

This seems to drive straight at those students who are not math majors but taking required math courses.

4. Problem-solving techniques

I summarize the problem-solving techniques discussed in Pólya, 2004. Pólya gives four steps:

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

As I discussed earlier, this method can sometimes feel linear, and as students often report, it feels as if one needs a masterplan before starting the search for an answer. Such criticism is echoed in Lesh and Zawojewski (2007):

One interpretation of Pólya's heuristics is that the strategies are intended to help problem solvers think about, reflect on, and interpret problem situations, more than they are intended to help them decide what to do when "stuck" during a problem attempt.

I instead offer an unordered list of tips that can give students ways to consider a problem and then methods to try and find the solution:

- *Do something!* Draw a picture, rewrite the instructions, make notes, just get your mind busy with the problem.
- *Keep it simple!* Do the simplest possible things first.
- *Find a partial solution.* It might lead to a full solution and it has value on its' own.
- *Give yourself a chance!* Your brain needs time to process information, don't give up if it doesn't seem obvious right away.
- *There is no light bulb!* There will probably not be a flash of inspiration, light bulb over the head moment. The way to the solution often just tumbles out as you make sense of everything.
- *Guess!* You might be lucky, but as long as you evaluate the merit of your guess you will certainly learn more about the problem and be able to perhaps make a better guess.
- *Backtrack, don't start from scratch.* At what point did things go wrong? Could you have made a different choice?
- *Be critical!* If you think you have an answer then be critical of it. Does it stand up? Is there a way to test it? Could there be other possible answers? If you don't have an answer, then are you interpreting the conditions correctly? Is there something wrong with your approach?
- *Reevaluate!* Could you reinterpret the question? Does it still make sense once you have a "solution"?

To emphasize the non-linear nature of problem-solving, I illustrate the techniques with the following diagram:

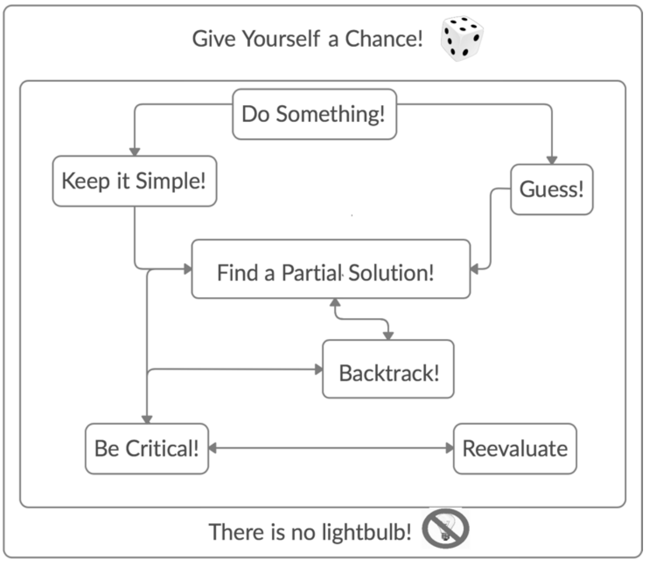


Figure 1. Non-linear nature of problem-solving

Students must know that they have options when problem-solving. From my experience, students who can persist and take alternative approaches when solving problems have more success and satisfaction in the process.

5. Puzzle case studies

I consider some specific puzzles and discuss how they could be taught within a class. When selecting puzzles to use with a class, I look for puzzles that mirror similar problem-solving techniques to the mathematical problems that will be presented or puzzles that can introduce the mathematical concept itself. The following are 5 puzzle case studies along with additional puzzles that are (loosely) of a similar theme. All of these are puzzles that I have used in class many times. Note that I have given references to the specific books that I have accessed the puzzles from, but they are all certainly available from many different sources.

5.1. Puzzle 1 - Definition puzzles (Gardner, 2001)

This is a good puzzle to use right at the start of a course as it gives students a sense of the nature of puzzles - that is, trying to understand what the puzzler had in mind when they were writing the puzzle. Indeed, as the literature suggests, it is often the metacognitive aspects of problem-solving that require the greatest attention.

Which is the odd one out?

Figure 2. Puzzle 1

This puzzle is a beneficial puzzle to solve as a group using a kind of Socratic method to model how students could try to ask themselves questions when problem-solving.

Do something! and *Guess!* are good tips for this puzzle. Hopefully, students will come up with 4 possible solutions related to the unique features of all but the first shape.

Now comes *Be critical!* and *Reevaluate!* Students should be pushed to recognize that there cannot be 4 solutions, but each of the "solutions" suggests a way to reinterpret the question. That is, that the puzzler defined "odd one out" in terms of unique properties.

Only the first one has no unique properties. This is a good example of *There is no light bulb!* as the answer just presented itself even if it took a little reevaluation of the question to see that it really is the desired solution.

This puzzle could be grouped under the title of "definition puzzles" - that is, puzzles that require the puzzler to analyze what is being asked very carefully. As such, it is a useful puzzle to use when prompting students to formulate a definition. For example, I have used this as a precursor to having students ’create’ a definition of a function. Similar puzzles would be:

- Any kind of rebus puzzles Designs, 2017, these are especially good for discussing notation with students.
- "Where in the world?" Puzzle 8.7 in Posamentier, 2003.

5.2. Puzzle 2 - Occam’s razor puzzles (Loyd, 1976)

This puzzle is an adaptation of a Sam Loyd puzzle, see the full version in Loyd (1976).

Can you connect the houses with their correspondingly numbered
gates by roads that do not cross?

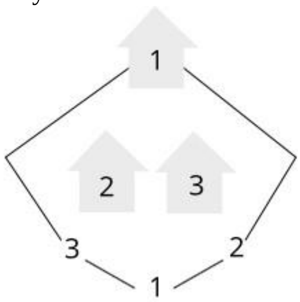
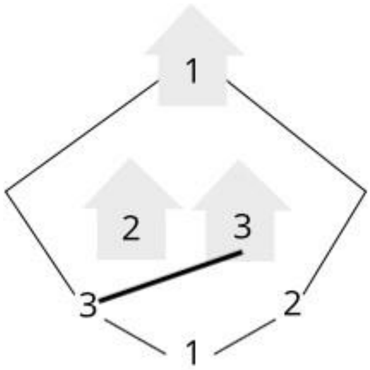


Figure 3. Puzzle 2

This puzzle is an excellent example of using *Do something!*, *Keep it simple!* and *Find a partial solution.* First have students make it work for one house. Can they extend the solution to a second? From that to all three houses? If not, then make sure to use *Backtrack, don't start from scratch.*

So for example, they might use *Keep it simple!* and manage to do one house as shown in image (a). From this, they might try to extend to image (b). Upon trying to extend again, they will hopefully see that there is no way to do so.

Therefore, this is a time for *Backtrack, don't start from scratch.* They could try instead what is shown in image (b).



(a) (b) (c)

Figure 4. Development of a solution to Puzzle 2

They would then recognize that there is really no option but for the final road to wind around the houses as follows:

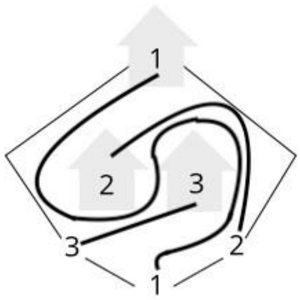


Figure 5. Puzzle 2

This is again an example of *There is no light bulb!*. This puzzle could be grouped in with other puzzles where the answer just 'tumbles-out'. That is, by applying Occam's razor Ariew, 1976 - doing the simplest thing at each step - the answer just tumbles out. These types of puzzles are particularly useful for working with solving equations. Encouraging students to take each step at a time and to do the simplest things first (divide out a constant for example).

Similar puzzles would be:

- "Logical thinking" puzzle 3.5 in Posamentier (2003).

- "The prisoner's dilemma" on page 182 of Solomon (2016).
- "Drive your car around the block" from August 26th of Bellos, n.d.
- "Knights and knaves" from Smullyan (1981).

5.3. Puzzle 3 - Check your assumption puzzles (Loyd, 1976)

This is known as the nine dots problem and appears in Loyd (1976). It is (somewhat) famous for allegedly being the source of the cliché: "Think outside the box."

Can you draw 4 straight lines without taking your pen off the paper
so that you go through all 9 dots?

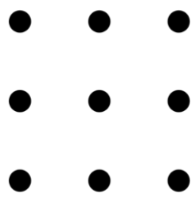


Figure 6. Puzzle 3

Puzzle 3 is all about *Do something!*. If students don't start drawing some lines and get suitably familiar with the problem, it is unlikely that they will be able to find the solution. Almost all students, and myself included, will stay within the grid at the beginning. This is when *Be critical!* comes in. If I can't find a solution despite trying lots of things, what could I do differently? In this case, students should get to the point of trying to leave the grid. (Or thinking outside the box). The solution should then present itself.

This example can be thematically grouped with other puzzles where all assumptions must be fully addressed. Students have often reported to me that they got a word-problem wrong because they over-thought it and didn't fully grasp the parameters; it is with such problems that this puzzle is a useful tool. Similar problems are:

1. "The bridges of Königsberg" on page 96 of Solomon (2016).
2. The birthday problem", puzzle 7.4 of Posamentier (2003).
3. "Rope around earth", puzzle 5.7 of Posamentier (2003).
4. Fermi (estimation) puzzles. Weinstein and Adam (2009).

5.4. Puzzle 4 - Introducing math concept puzzles)

This next puzzle is a terrific introduction to binary code in computing, but I approach it in class as simply a puzzle in and of itself.

You have 4 envelopes and 15 $1 bills. Can you place the bills in the envelopes in such a way that you can still pay any dollar amount by handing over one or more of the envelopes?

Here is another fine example of *Keep it simple!* and *Find a partial solution.* That is, can I put some bills in envelopes to first pay $1? What about then $2? $3? and so on. In this way, students should see again that *There is no light bulb!*. The first envelope must have just one bill in it, the second then will have two, the third four, and the fourth eight.

This puzzle therefore naturally introduces a geometric sequence or an exponential function. There are many puzzles which introduce math concepts - indeed a paper on puzzles that do this would be very interesting - but I list just a few of them here:

- "Thou shalt not divide by zero", puzzle 6.3 of Posamentier, 2003.
- "Area of a circle", puzzle 5.20 of Posamentier, 2003.

5.5. Puzzle 5 - Guess and check puzzles (Gardner, 2001)

Finally, the following puzzle is very much a *Guess!* and *Be critical!* puzzle:

Fill the grid with the numbers 1 through 8 so that consecutive numbers
do not touch - neither edges nor corners.

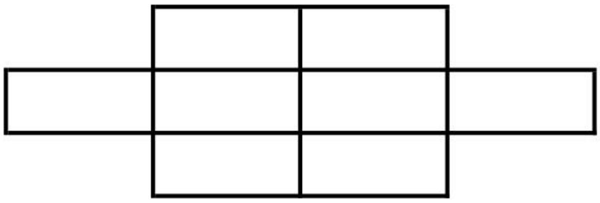


Figure 7. Puzzle 5

Students should try to *Guess!* to start with - every student can put 8 numbers into a grid - and maybe some will get lucky. If not, then try again, at least 3 or 4 times.

Remember to *Give yourself a chance!* Then comes the time to *Be critical!*. Was there anything they learned? Did some numbers behave differently than others? With a little time, students should recognize that 1 and 8 only have one consecutive number each to worry about. Most will then recognize that the best place for them is in the two middle spots. From that point forward, 2 and 7 have only one place each to go. Finally, with four numbers and four spots, students should feel emboldened to *Guess!* and then *Backtrack, don't start from scratch!* at this point.

Puzzle 5 is a good example of puzzles where there is only a finite number of possibilities. With enough patience, the student can run through enough possible solutions to get the true solution. In my experience, a patient approach is an important tool to teach students; indeed many equations could be solved (or at least estimated) in this guess and check manner. Similar puzzles include:

- "Cat and dog race" from page 16 of Loyd, 1976.
- "Magic/unmagic squares" from page 130 of Gardner, 1967.
- "Monty Hall problem", puzzle 7.6 of Posamentier, 2003.
- "Dissection dilemma", puzzle 57 of Gardner, 1994.
- "Counterfeit coins", puzzle 9 of Gardner, 1994.

6. Course integration

The following course integration is intended to complement and to reinforce existing course structures, not to replace them. In fact, I suggest merely adding a puzzle to the start of class and using a course-specific problem that uses similar problem-solving tips at the end of class.

This approach is in line with the recommendation of Meyer III et al. (2014):

> *...we would recommend initiating a teaching experience in Puzzle-based Learning in a limited setting such as an outreach effort or as a teaching tactic in another course.*

I split into two considerations: how a sample class period could look in addition to how the overall course could be structured to involve puzzles and problem-solving techniques.

6.1. A sample class period

Puzzles lend themselves very well to the start of class, and in this context are often known as ice breakers. They certainly serve this purpose, but I focus here on the development of problem-solving techniques. The use of general problem-solving at the start of the class can then be refined to a more particular, course-specific problem later on.

It is important here that the puzzle selected will mirror solving techniques that will be required for the curriculum problem posed at the end of the period. There is a danger that students might feel fatigued by the puzzles if they seem randomly selected and not linked to the material or problem-solving technique being utilized that period. In my experience, it is very important to emphasize **how** the puzzle will relate and then again emphasize the connection at the end of the class.

The following is a sample class that I taught last year to a precalculus class:

Start of class period

Puzzle 1 is on the board or screen as students enter, and they can begin to consider the problem and switch to problem-solving mode immediately. Expectations should be set early on that classes start this way, and the students should begin working on the puzzle right away.

Recall that puzzle 1 was a good example to start with *Do something!* and *Guess!* before finishing with *Be critical!* and *Reevaluate!*.

This exercise should take no more than five minutes of class time, given that students can work on it before class, too. They must try the puzzle, and they should be reminded that a partial solution is okay.

<u>End of class period</u>

This part of integration is designed to fully engage Lester's *complexity principle*, placing problem-solving in the context of developing mathematical understanding.

I use the example of systems of equations where methods of solution are introduced during the class period. Students should then be presented with a problem that generalizes the topic presented. In this case, an example such as:

A business calculates its' fixed costs at $10,000 per month and its' variable costs at $25 per item produced. Each item sells for $100. Can you calculate the number of items they must sell to make a profit?

Students should work in pairs or small groups - engaging *the group interaction principle*. They should be reminded of the puzzle at the start and given the tips *Do something!* and *Guess!*. In the context of this particular problem, advise them against diving straight into systems of equations and encourage them to try and guess without setting it up as a mathematical problem.

Then once students are familiar with the problem, they can construct it as a system of equations and solve using the methods given.

Finally, make sure to remind students about *Be critical!* and *Reevaluate!*. In solving the particular system of equations they may have lost sight of the scenario. Does their solution make sense? Is there any way to check their answer? How do they know it is true?

During this period, the instructor will be fully engaging the *multiple roles for the teacher principle* by having already modeled good problem-solving with the puzzle; by monitoring the progress of each group; and by prompting students to use specific problem-solving tips.

Students should then hand in their solutions, that is, well-presented work that shows how they arrived at a solution and why it is the correct one. This work can be graded as a quiz or simply for participation. In any case, it should be low stakes to encourage students to approach the problem with more freedom, but collecting work also engages the *assessment principle*.

In my experience, this activity should last 10 minutes, though it will vary upon the class and the topic.

6.2. Overall course integration

In determining how puzzles and problem-solving fit in as a part of the overall course structure, I refer to Lester's principles. In particular, *the prolonged engagement principle* and *the task variety principle* are crucial considerations. With this in mind, I suggest that most classes should have some explicit reference to problem-solving methods, whether in the form of puzzles as the sample class period above or some other method. In fact, *the task variety principle* suggests that alternative "problematic tasks" are important.

Instructors should be emboldened to consider activities that encourage students to solve problems in any context, practical or theoretical.

The systematic organization principle makes it clear that problem-solving instruction should be explicit. That is, make it obvious to students when a *general* problem-solving method is being presented as opposed to a particular method of solving a specific type of problem. I recommend that the syllabus should explicitly state this as a learning objective. As such, the following is an example I have used of how the problem-solving objective could be adapted:

Students will develop new and enhance existing problem-solving skills *by*:

1. regularly working on problems
2. working on a variety of problems
3. learning general problem-solving techniques alongside mathematical content
4. working within groups
5. being assessed and given feedback on good use of problem-solving methods.

The last of these objectives will require some attention. In particular, the teacher must consider how to assess "good use of problem-solving methods". I propose to do this by regularly asking students to reflect on how they attempted problems. Reflection is an opportunity for students to develop their metacognitive awareness - judge their problem-solving skills - and to appreciate what logical and yet playful processes they are developing through mathematics.

For example, at the end of the sample class period given in 7.1, I tasked students with the following:

> Reflect upon how you attempted to solve the problem. In particular, which of the problem-solving techniques did you use? Can you give a short narrative of how you attempted the problem? Could you have solved it in a differently? What value did this problem have?

In order to assess the work students hand in, there is a companion to the solution that allows some judgment of "good use of problem-solving methods." A rubric that I have used for grading such a reflection follows the model of Hatton and Smith (1995):

1. Descriptive: The student simply restates their attempt to solve the problem, and there is no mention of why they attempted to solve in that way.

2. Descriptive Reflection: The student gives some techniques that they used, but do not give any explanation of why they used those techniques.

3. Dialogic Reflection: The student gives some thought to why they used a technique and questions their use and considers alternatives. Students might use statements such as "I wonder..., what if..., perhaps...", etc.

4. Critical Reflection: The students fully questions all choices and decisions they made, gives possible alternative approaches to solving the problem, including why they chose to not use an approach. They engage in reflective questioning, that is, some broader questioning of the value of the question. For example, they might question whether it is realistic if it is an applied problem. They may even attempt a refinement of the problem.

Notably, the rubric does not necessarily require the student to have solved the problem. I posit that mathematics classes should begin to have a broader understanding of the value of mathematics - particularly in the case of non-major students taking lower-division classes as a requirement. In this case, our goal for these students should involve developing their sense of the value of mathematical thinking. I look at this goal as almost in the mold of a music appreciation or poetry course, where the objective is for students to see the broader value of these disciplines.

7. Puzzle resources

I have resources for sourcing mathematical puzzles. This list is not exhaustive, simply those that I have utilized most often:

1. Sam Loyd’s Cyclopedia of 5,000 Puzzles, Tricks and Conundrums Loyd (1976).

2. The scientific American book of mathematical puzzles and diversions by Martin Gardner Gardner (1967).

3. The colossal book of mathematics: classic puzzles, paradoxes, and problems: number theory, algebra, geometry, probability, topology, game theory, infinity, and other topics of recreational mathematics by Martin Gardner Gardner (2001).

4. My best mathematical and logic puzzles by Martin Gardner Gardner (1994).

5. Guesstimation: Solving the world’s problems on the back of a cocktail napkin by John Adam and Lawrence Weinstein Weinstein and Adam (2009).

6. What is the name of this book? The riddle of Dracula and other logical puzzles by Raymond Smullyan Smullyan (1981).

7. The big book of rebus puzzles by Zentopia Designs Designs (2017).

8. Alex Bellos’ Monday Puzzle column in the Guardian Bellos, n.d.

9. Math charmers: Tantalizing tidbits for the mind by Alfred S. Posamentier Posamentier (2003).

10. Robert Solomon’s Little book of mathematical principles, theories, and things Solomon (2016).

8. Summary

The goal of this paper is to help teachers give students a better understanding of the more creative, playful, and game-like qualities of mathematics; as well as to enhance the existing mathematical curriculum by developing more general problem-solving skills. I presented the reasons for attempting this goal, as well as the specific problem-solving tips that can be offered to students. I discussed exposition of the literature on problem-solving, including specific ways to implement the literature in an undergraduate mathematics class.

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