

Organisation and Integration of Procedural and Conceptual Knowledge in Geometric Reasoning across the van Hiele Levels

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Abstract: This study examines how procedural and conceptual demands are organised across the five van Hiele levels of geometric reasoning and develops a theoretical account of their relationship. Using qualitative document analysis and directed content analysis, 25 representative descriptors of Visualization, Analysis, Abstraction, Deduction, and Rigor were synthesized from established accounts of the van Hiele framework and analysed through four dimensions: the role of procedural or operational activity, conceptual demand, the form of procedural-conceptual coordination, and the epistemic basis of reasoning. The analysis indicates a developmental transformation rather than a simple replacement of procedures by concepts. At Visualization, reasoning is mainly perceptual-operational. At Analysis, procedural activity supports attention to properties. At Abstraction, actions, representations, and informal arguments are explicitly coordinated with relational reasoning. At Deduction and Rigor, procedural activity remains present but becomes increasingly embedded within formal conceptual and axiomatic structures. These patterns are synthesized as the Procedural-Conceptual Integration Trajectory (PCIT), which describes progression from perceptual-operational engagement through property-based and relational coordination to deductive and axiomatic regulation. The PCIT provides a conceptual bridge between van Hiele theory and research on conceptual-procedural knowledge and offers a basis for designing geometry tasks that progressively connect action, representation, explanation, justification, and proof.

Keywords: van Hiele levels, geometric reasoning, procedural knowledge, conceptual understanding, geometry instruction.

1. Introduction

1.1. Background

Geometric reasoning is central to mathematics because it supports spatial thinking, representation, logical argument, proof, and problem solving. Learners, however, do not move automatically from recognising figures to reasoning about properties, relations, definitions, and axiomatic systems. The van Hiele model remains one of the most influential frameworks for explaining this development because it describes qualitatively different levels of geometric thought and emphasises the role of instruction in supporting progression (Crowley, 1987; Fuys et al., 1988; van Hiele, 1986).

In the five-level numbering adopted in this paper, geometric reasoning progresses from Visualization (Level 1), through Analysis (Level 2) and Abstraction (Level 3), to Deduction (Level 4) and Rigor (Level 5). At the lower levels, learners identify and describe figures largely through appearance and isolated properties; at intermediate levels they recognise relationships among properties and formulate meaningful definitions; and at higher levels they reason formally with theorems, proofs, axioms, and mathematical systems (Crowley, 1987; Moru et al., 2021; Usiskin, 1982; Vojkuvkova, 2012).

Conceptual knowledge refers to an interconnected understanding of mathematical ideas, relations, representations, and meanings, whereas procedural knowledge concerns knowledge of mathematical procedures and the conditions governing their appropriate use (National Research Council, 2001). These forms of knowledge are not independent; rather, research suggests that they can develop iteratively, with growth in one supporting growth in the other (Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001; Ndungo et al., 2025). Star (2005) further cautions against reducing procedural knowledge to rote execution, arguing that procedures may be understood and used with different levels of sophistication. This relationship is particularly important in geometry, where learners may construct, transform, measure, draw, classify, or manipulate representations while also reasoning about properties, relationships, definitions, and justification. The instructional challenge is therefore not to choose between procedural and conceptual approaches, but to understand how procedural activity and conceptual reasoning can be coordinated at different stages of geometric development. This question also has contemporary curriculum relevance. Uganda's Lower Secondary Mathematics Syllabus places emphasis on mathematical concepts, skills, critical thinking, problem solving, and application, reflecting the country's competency-based curriculum orientation (National Curriculum Development Centre [NCDC], 2019). Similar international trends are visible in recent comparative curriculum work.

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The OECD's analysis of mathematics curricula across countries documents growing emphasis on competencies such as problem solving, critical thinking, reasoning, data literacy, and digital literacy alongside mathematical content (Organisation for Economic Co-operation and Development [OECD], 2024). Thus, understanding how action, concepts, reasoning, and justification interact within a developmental theory such as van Hiele is relevant beyond the Ugandan context.

Despite substantial research on the van Hiele levels and on conceptual-procedural knowledge separately, the relationship between these two theoretical traditions has not been made sufficiently explicit. Existing descriptions of van Hiele levels identify increasingly sophisticated forms of geometric thought, but they do not directly explain how the role of procedural activity changes as conceptual reasoning develops. Conversely, research on conceptual and procedural knowledge establishes their interdependence but does not specify how that interdependence may be organised across the developmental structure of geometric reasoning. This leaves an important theoretical and instructional question: as learners move across van Hiele levels, are procedures replaced by concepts, merely added to concepts, or progressively reorganized within increasingly conceptual forms of reasoning? To address this gap, the study develops the Procedural–Conceptual Integration Trajectory (PCIT) as a theoretical account of how procedural and conceptual knowledge become progressively coordinated across the five van Hiele levels of geometric reasoning.

1.2. Problem Statement

Learners frequently experience difficulty moving from recognition and property identification toward relational reasoning, proof, and abstraction in geometry. The van Hiele model describes this progression, while research on mathematical proficiency emphasises the interdependence of procedural and conceptual knowledge. However, the two perspectives are rarely connected in a way that explains how procedural activity is transformed across levels of geometric reasoning.

Although previous descriptions of the van Hiele levels provide rich accounts of progressively sophisticated geometric thought, translating those descriptions into simple counts of 'procedural,' 'conceptual,' or 'dual-skill' characteristics is problematic. Some descriptors identify capabilities, whereas others identify boundaries or limitations of reasoning; treating them as equally weighted quantitative observations can therefore create false precision. What is needed is a qualitative developmental analysis that examines how procedural activity functions within increasingly sophisticated conceptual organisation rather than interpreting the descriptors as learner scores or population estimates.

The main research question guiding the study was: How are procedural and conceptual forms of mathematical knowledge organised and integrated across the five van Hiele levels of geometric reasoning?

The study was further guided by the following subquestions: (1) What are the defining characteristics of the five van Hiele levels when examined through a theory-driven qualitative framework? (2) How do procedural activity, conceptual demand, and their coordination change across the five van Hiele levels? (3) What developmental interpretation can be derived from these patterns to inform geometry instruction and future empirical research?

1.3. Theoretical and Conceptual Framework

Van Hiele Theory of Geometric Reasoning

The van Hiele theory originated in the work of Pierre van Hiele and Dina van Hiele -Geldof and was later elaborated and applied widely in geometry education. The theory proposes qualitatively different levels of thought rather than a simple accumulation of content. Learners at one level interpret geometric objects through structures that differ from those available at the next level (van Hiele, 1986). Instruction and language are therefore central to progression.

At Visualization, figures are recognised primarily by overall appearance and prototypes. At Analysis, attention shifts to properties of figures, although relations among properties may remain weak. At Abstraction, learners coordinate properties, recognise class inclusion, formulate definitions, and engage in informal deduction. At Deduction, reasoning becomes organised within formal systems of definitions, theorems, necessary and sufficient conditions, and proof. At Rigor, learners reason about axiomatic systems themselves and can compare alternative systems such as Euclidean and non-Euclidean geometries (Crowley, 1987; Fuys et al., 1988; Usiskin, 1982).

Conceptual Understanding and Procedural Knowledge

Throughout this study, conceptual knowledge and procedural knowledge are used as the principal theoretical terms. Conceptual knowledge refers to interconnected knowledge of mathematical concepts, principles, relationships, and representations, whereas procedural knowledge refers to knowledge of mathematical procedures and the conditions governing their appropriate use. The term conceptual demand refers to the

conceptual knowledge required by a descriptor, while procedural activity refers to the mathematical actions implied by a descriptor, such as constructing, measuring, transforming, or manipulating representations. Conceptual reasoning refers to the use of conceptual knowledge in explaining, relating, justifying, or proving mathematical relationships.

Conceptual understanding is more than possession of isolated facts; it involves connected knowledge of mathematical ideas, relations, representations, and reasons, while conceptual knowledge more broadly concerns the relationships and principles that connect mathematical ideas (Baroody et al., 2007; Crooks & Alibali, 2014). Procedural fluency, in contrast, concerns knowledge and execution of mathematical procedures, including knowing when and how procedures are appropriately used (National Research Council, 2001). In geometry, procedures may include constructing figures, transforming objects, measuring, drawing auxiliary lines, or operating on representations.

The distinction is analytically useful but should not be treated as a strict dichotomy. Research has increasingly emphasised the interrelationship between conceptual and procedural forms of mathematical knowledge rather than treating them as mutually exclusive categories (Barumbun & Kharisma, 2022; Rittle-Johnson et al., 2001). Star (2005) similarly argued that procedural knowledge may vary in depth and flexibility. These perspectives support an interpretation in which the role of procedure may change as reasoning develops: procedures can be perceptually driven, exploratory, relationally coordinated, or embedded within formal proof.

Analytical Proposition: From Coexistence to Integration

The study therefore adopts the proposition that the important developmental question is not whether a Van Hiele level is simply 'procedural' or 'conceptual,' but how procedural activity is related to conceptual organisation at that level. This shifts the analysis from categorical replacement to functional integration. The resulting framework examines four dimensions: the role of procedural or operational activity, the depth of conceptual demand, the form of procedural-conceptual coordination, and the epistemic character of reasoning.

2. Methodology

2.1. Research Design

In this study, a qualitative approach based on document analysis was employed. Document analysis is appropriate when texts or formal descriptions constitute the data from which meaning, patterns, and theoretical relationships are interpreted (Bowen, 2009). Because the initial analytical categories were informed by existing theory on conceptual and procedural knowledge, the study used a directed qualitative content-analysis approach (Hsieh & Shannon, 2005).

The study was not an empirical test of learners' van Hiele levels and did not analyse scores from a geometry achievement instrument. Its data consisted of theoretical descriptors of the five van Hiele levels. Consequently, the findings concern the cognitive demands represented in the framework rather than the prevalence of particular forms of knowledge among learners.

2.2. Data Corpus and Unit of Analysis

The analytical corpus consisted of 25 representative descriptors of the five van Hiele levels, with five descriptors used to characterise each level. The descriptors were synthesised from recurring features reported in foundational accounts of the van Hiele framework, particularly van Hiele (1986), Usiskin (1982), Crowley (1987), and Fuys et al. (1988), together with subsequent applications and analyses of the framework, including Abdullah and Zakaria (2013), Malatjie and Machaba (2019), Mbusi and Luneta (2021), Ramírez-Uclés and Ruiz-Hidalgo (2022), Senk et al. (2022), Arnal-Bailera and Manero (2024), Mahfudy and Afriana (2024), Putri et al. (2024), and Satır and Kurtuluş (2024). These sources constituted the documentary corpus from which recurring descriptions of reasoning at the five levels were identified and synthesised. Selection sought to preserve the defining cognitive character of each level while incorporating both positive capabilities and important boundaries of reasoning. The individual descriptor was the initial unit of analysis; however, interpretation was conducted primarily at the level-profile because the descriptors are theoretically related rather than independent quantitative observations.

This level-profile approach was necessary because the descriptors do not function as equivalent test items. For example, recognising relationships among properties describes an available form of reasoning, whereas not yet recognising such relationships describes a boundary of reasoning. Both are theoretically informative, but counting them as equivalent 'skills' would distort their meaning. The descriptors were therefore retained as textual evidence and interpreted through their function within each level. Table 1 below shows the distribution of the indicators identified in the study according to the van Hiele levels.

Table 1. *Analytical corpus of representative van Hiele descriptors*

Van Hiele level	Synthesized descriptor used in the analysis
Level 1: Visualization	Recognises geometric figures primarily by overall appearance rather than by defining properties.
Level 1: Visualization	Relies strongly on visual perception and non-verbal or prototype-based reasoning.
Level 1: Visualization	Treats a figure as a whole and compares it with familiar prototypes.
Level 1: Visualization	Can sort, name, or categorize figures using simple descriptive language.
Level 1: Visualization	Does not yet use mathematical properties systematically as the basis for class inclusion or definition.
Level 2: Analysis	Identifies and names properties of geometric figures.
Level 2: Analysis	Examines sides, angles, parallelism, congruence, and other attributes through empirical activity.
Level 2: Analysis	Treats properties individually and may regard them as equally important.
Level 2: Analysis	Uses observation, measurement, drawing, or construction to discover geometric facts.
Level 2: Analysis	Does not yet coordinate relationships among properties or require formal proof.
Level 3: Abstraction	Recognises relationships among properties and among classes of figures.
Level 3: Abstraction	Formulates meaningful definitions based on relationships among properties.
Level 3: Abstraction	Provides informal arguments or explanations to justify geometric relationships.
Level 3: Abstraction	Uses sketches, diagrams, grids, constructions, or transformations to explore relationships.
Level 3: Abstraction	Engages in informal deduction and class-inclusion reasoning.
Level 4: Deduction	Constructs and understands formal deductive proofs.
Level 4: Deduction	Distinguishes necessary from sufficient conditions.
Level 4: Deduction	Identifies implications and relationships among properties, definitions, and theorems.
Level 4: Deduction	Understands the roles of definitions, axioms, theorems, and proof within a deductive system.
Level 4: Deduction	Uses formal reasoning to validate relationships and solve complex geometric problems.
Level 5: Rigor	Reasons about mathematical or axiomatic systems as objects of study.
Level 5: Rigor	Compares Euclidean and non-Euclidean geometric systems.
Level 5: Rigor	Examines consequences of adding, removing, or altering axioms.
Level 5: Rigor	Engages in abstract generalization across geometric systems.
Level 5: Rigor	Works with advanced proofs and the structure and consistency of axiomatic systems.

Note. The descriptors are synthesized representations rather than verbatim statements from individual sources. They were developed through comparison of recurring descriptions of the van Hiele levels across van Hiele (1986), Usiskin (1982), Crowley (1987), Fuys et al. (1988), Abdullah and Zakaria (2013), Malatjie and Machaba (2019), Mbusi and Luneta (2021), Ramírez-Uclés and Ruiz-Hidalgo (2022), Senk et al. (2022), Arnal-Bailera and Manero (2024), Mahfudy and Afriana (2024), Putri et al. (2024), and Satir and Kurtuluş (2024).

2.3. Directed Coding Framework

A four-dimension coding framework was developed to guide analysis. The framework examined (a) the procedural or operational role of mathematical action, (b) the conceptual demand, (c) the form of procedural-conceptual coordination, and (d) the epistemic basis on which a geometric claim is recognised or justified. The dimensions and decision rules are presented in Table 2. The framework does not assign learner scores; it identifies the dominant function represented in each descriptor and supports synthesis of the level as a whole.

Table 2. *Directed coding framework and decision rules*

Dimension	Analytical question	Indicative evidence
Procedural/operational role	What kind of mathematical action is expected or implied?	Recognition, sorting, construction, measuring, drawing, transforming, manipulating representations, applying an established sequence.
Conceptual demand	What form of meaning or relationship must be understood?	Properties, relationships among properties, class inclusion, definitions, necessary/sufficient conditions, theorem structure, axioms, generalization.
Integration form	How are action and conceptual reasoning related?	Weak/separate; emerging support; explicit coordination; conceptually governed; systemically embedded.
Epistemic character	What counts as a valid basis for knowing at this level?	Appearance/prototype; empirical property inspection; relational/informal justification; formal deduction; axiomatic/system-level reasoning.

2.4. Analytical Procedure

Analysis proceeded in three stages. First, each descriptor was read for its implied mathematical action, conceptual object, and basis for justification. Second, descriptors within the same van Hiele level were compared to build a coherent level profile across the four coding dimensions. Third, the five profiles were compared sequentially to identify changes in the role of procedure, depth of conceptual organisation, and form of coordination across levels.

An analytic memo was maintained for each level to distinguish what is explicitly stated in the van Hiele descriptors from the interpretation developed in this study. Particular attention was given to negative descriptors and to the possibility that procedural activity continues at higher levels even when it is no longer the defining feature of reasoning. This prevented the analysis from equating 'not coded as procedural-only' with 'no procedures occur.'

No inferential statistical test was used because the corpus is theoretical, small, and non-random. Likewise, no percentage is interpreted as an estimate of learner performance. Where counts are mentioned, they are descriptive of the source corpus only and are secondary to the qualitative interpretation.

2.5. Trustworthiness and Analytical Caution

Credibility was strengthened through explicit coding rules, comparison with multiple foundational accounts of the van Hiele framework, an audit trail linking interpretations to descriptors, and deliberate consideration of alternative interpretations. In particular, the analysis treated the absence of a procedural-only descriptor as different from the absence of procedural activity. Because the descriptors and coding framework were interpreted without an independent second subject-matter coder, consistency of the coding across independent judges could not be estimated. This limits claims regarding coding reliability. Accordingly, the PCIT is presented as a theoretically derived analytical proposition rather than a validated classification system and requires independent coding and empirical testing in future research.

2.6. Ethical Considerations

The study used published theoretical and curricular documents and did not involve recruitment of human participants or analysis of identifiable personal data. Ethical responsibility therefore centred on accurate representation of sources, transparent analytical procedures, appropriate citation, and avoidance of claims beyond what the documentary evidence can support.

3. Results

The analysis revealed a developmental transformation in the relationship between procedural activity and conceptual reasoning. Rather than a linear shift in which procedures disappear and concepts replace them, procedural activity changes function: it is initially visible and primary, later supports conceptual inspection, becomes explicitly coordinated with relational reasoning, and finally operates within increasingly formal conceptual structures. Table 3 summarises the cross-level profiles generated through the directed analysis.

Table 3. *Cross-level procedural-conceptual profiles and PCIT phases*

Van Hiele level	Procedural/operational role	Conceptual demand	Integration form	Interpretive phase
Level 1: Visualization	Primary and visible: recognition, comparison, sorting, prototype use	Limited/implicit; properties are not yet the main basis of identification	Weak coordination	Perceptual-operational engagement
Level 2: Analysis	Supports inspection, naming, measuring, and description of properties	Emerging analytic understanding of individual properties	Emerging coordination	Property-based analysis
Level 3: Abstraction	Representations, sketches, constructions, and other actions support relational inquiry	Relational understanding; definitions, class relations, informal justification	Explicit coordination	Relational coordination
Level 4: Deduction	Embedded within proof construction and formal argument	Formal deductive understanding of theorem structure, implication, and necessary/sufficient conditions	Conceptually governed	Deductive regulation
Level 5: Rigor	Embedded within work on formal systems and comparisons among systems	Axiomatic and system-level understanding	Systemically embedded	Axiomatic regulation

Level 1: Perceptual-Operational Engagement

At Visualization, geometric figures are recognised largely through overall appearance and familiar prototypes. Sorting, comparing, and naming can occur, but mathematical properties are not yet the primary basis for identification. The procedural dimension is therefore best described as perceptual-operational rather than as mature procedural fluency. Learners act on and classify visual objects, yet the conceptual network linking properties, definitions, and classes remains limited.

The defining pattern at Level 1 is therefore visible action and recognition with weak procedural-conceptual coordination. Conceptual information may be present, but it does not yet organise recognition in a systematic property-based way. The level is better characterised by the function of action and perception than by a numerical proportion of procedural characteristics.

Level 2: Property-Based Analysis and Conceptual Emergence

At Analysis, attention shifts from the whole figure to its component properties. Learners can identify sides, angles, parallelism, congruence, and other attributes and can use empirical activity to investigate them. Procedural actions remain important because measuring, drawing, checking, and describing provide access to properties. However, the conceptual demand is stronger than at Visualization because the learner must attend to attributes that define or characterise figures.

Integration is therefore emerging rather than complete. Actions support conceptual noticing, but relationships among properties and hierarchical class relations are not yet fully coordinated. The transition from Level 1 to Level 2 is thus better interpreted as conceptual emergence within continuing operational activity, not as a simple numerical decline in procedure.

Level 3: Relational Coordination as the Integration Point

Abstraction represents the clearest point of explicit procedural-conceptual coordination. Learners recognise relationships among properties and classes, construct meaningful definitions, and provide informal arguments. Sketches, diagrams, grids, constructions, and transformations can be used not merely to produce an answer but to explore why relationships hold and to support justification.

This level therefore functions as a developmental bridge. Procedural activity is reorganised as a tool for relational reasoning. The important change is functional: a construction or representation is no longer only an action to complete; it becomes evidence within a conceptual argument. This pattern provides the strongest basis for the proposed Procedural-Conceptual Integration Trajectory.

Level 4: Conceptually Governed Procedure in Deduction

At Deduction, learners reason formally with definitions, theorems, axioms, implication, necessary and sufficient conditions, and proof. Constructing a proof nevertheless requires sequences of mathematical moves, representation choices, and inferential procedures. The important change is therefore not the disappearance of procedural activity, but the fact that such activity is governed by a formal conceptual structure.

The defining feature of Level 4 is therefore not procedural absence but conceptual regulation. Procedures are justified by their place in a deductive system. Knowing what step can be taken is inseparable from knowing why that step is warranted.

Level 5: Axiomatic Regulation and System-Level Reasoning

At Rigor, reasoning concerns mathematical systems themselves. Learners can compare Euclidean and non-Euclidean systems, consider the consequences of changing axioms, and reason at a level where definitions and theorems are understood relative to an axiomatic structure. Procedural activity is therefore even more deeply embedded within conceptual organisation.

The epistemic shift is from proving within a system to reasoning about systems. Operations, symbolic manipulations, and proof procedures may still occur, but their meaning depends on the axioms and structures being examined. The highest level thus reflects systemic integration rather than the disappearance of procedure.

The Procedural-Conceptual Integration Trajectory (PCIT)

Cross-level comparison produced the Procedural-Conceptual Integration Trajectory (PCIT). The trajectory captures two simultaneous developments: conceptual organisation becomes increasingly relational, formal, and systemic, while procedural activity changes from being visible and primary to being increasingly embedded within conceptual reasoning. Figure 1 presents the model.

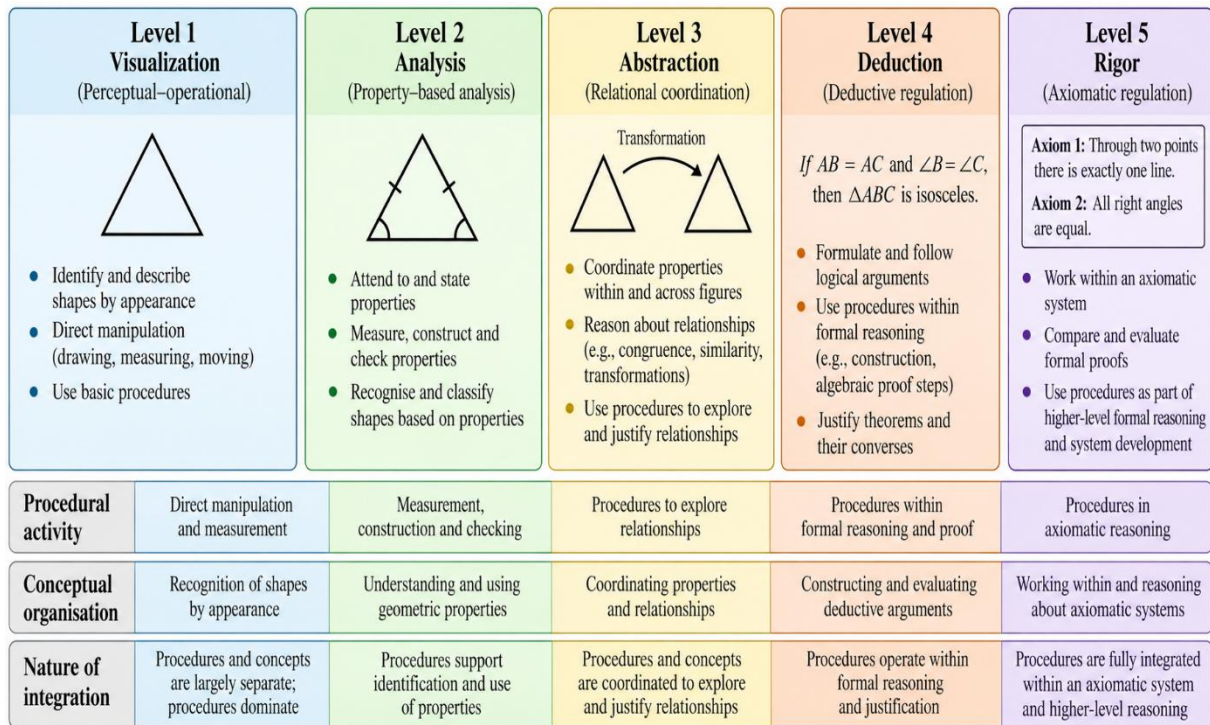


Figure 1. *Procedural-Conceptual Integration Trajectory across the five van Hiele levels.*

Figure 1 presents the Procedural–Conceptual Integration Trajectory (PCIT), developed by Ndungo and Akugizibwe, as a theoretical model illustrating how procedural and conceptual knowledge are progressively coordinated across the five van Hiele levels.

Procedural activity is initially visible and primary, supporting recognition, manipulation, and basic geometric action, but becomes increasingly embedded within property-based, relational, deductive, and axiomatic reasoning. At the same time, conceptual organisation becomes progressively more complex and abstract. The trajectory therefore represents not the disappearance of procedural knowledge, but its progressive reorganisation within increasingly sophisticated forms of geometric reasoning.

4. Discussion and Conclusions

4.1. Discussion

The findings of this study suggest that the relationship between procedural and conceptual knowledge across the van Hiele levels is better understood as a process of progressive reorganization than as the replacement of procedures by concepts. A simple procedural-to-conceptual interpretation would imply that lower levels of geometric reasoning are essentially procedural and that procedures become less important as conceptual sophistication develops. The analysis does not support such a separation. Instead, procedural activity remains evident across the five levels, but its role changes as conceptual organization becomes increasingly relational, deductive, and axiomatic. This interpretation is consistent with research showing that conceptual and procedural knowledge can develop iteratively and mutually support one another (Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001). It is also consistent with Star's (2005) argument that procedural knowledge should not be reduced to rote execution because procedures may be understood and used with different levels of sophistication. Empirical work in transformation geometry similarly suggests that procedural and conceptual development can follow related trajectories under van Hiele phased instruction rather than functioning as completely separate forms of learning (Ndungo et al., 2025).

The PCIT extends this argument specifically to geometric reasoning by proposing that what changes across the van Hiele levels is not simply the amount of procedural or conceptual knowledge available to a learner, but the way procedural activity is organized within increasingly sophisticated conceptual structures. The van Hiele framework describes progression from recognition of figures through analysis of properties, relational reasoning, formal deduction, and ultimately reasoning within axiomatic systems (Crowley, 1987; Fuys et al., 1988; van Hiele, 1986; Usiskin, 1982). The present analysis adds to this developmental account by making explicit the changing function of procedural activity within that progression. At Visualization, procedures are closely associated with recognition, comparison, sorting, and direct manipulation of figures. At Analysis, measuring, drawing, constructing, and inspecting representations support the identification and description of properties. At Abstraction, these activities become increasingly coordinated with relationships among properties, class inclusion, definitions, conjectures, and informal justification. At Deduction and Rigor, procedural activity becomes embedded within proof construction, logical implication, theorem structure, and reasoning about axiomatic systems.

The importance of this interpretation lies in the recognition that the same mathematical action may serve different cognitive purposes depending on the level of reasoning in which it is situated. Constructing a quadrilateral, for example, may at one level support recognition of a familiar shape, at another support identification of properties, and at a higher level provide evidence for a conjecture or justification of a class relationship. The procedure itself may appear similar, but its epistemic function changes. This perspective is compatible with the broader conception of mathematical proficiency proposed by the National Research Council (2001), which emphasizes the interdependence of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. From the perspective of the PCIT, the developmental issue is therefore not whether learners possess procedural or conceptual knowledge, but how these forms of knowledge are coordinated within increasingly sophisticated geometric reasoning.

A particularly important feature of the trajectory is the position of Level 3, Abstraction. Within the van Hiele framework, this level is associated with recognition of relationships among properties, hierarchical classification, meaningful definitions, and informal deduction (Crowley, 1987; Fuys et al., 1988). The present analysis suggests that its significance may also lie in the more explicit coordination of procedural activity with relational reasoning. At this level, a learner may use a construction, transformation, diagram, or representation not merely to inspect a figure but to identify invariants, compare classes, formulate a conjecture, refine a definition, or support an informal argument. The procedure therefore becomes a resource for reasoning rather than simply an action to be completed.

This interpretation provides a possible explanation for why movement from empirical activity to formal deduction can be difficult. Exposure to increasingly complex content may not be sufficient if learners are not supported to connect what they do with what they notice, explain, relate, and justify; making underlying mathematical relationships and structures explicit is important for developing conceptual understanding (Richland et al., 2012). The PCIT therefore suggests that progression toward higher levels of geometric reasoning may depend partly on the quality of procedural-conceptual coordination developed at intermediate levels. This proposition is consistent with empirical evidence from transformation geometry showing that structured van Hiele phased instruction can support learners' progression across levels of geometric reasoning (Ndungo et al., 2026). Such evidence does not itself validate the PCIT, but it reinforces the importance of examining how instructional organization may influence movement across van Hiele levels.

The PCIT should consequently be understood as an interpretive extension of the van Hiele framework rather than as a replacement for it. The van Hiele model identifies qualitatively distinct levels of geometric thought, whereas the PCIT focuses specifically on the changing relationship between procedural activity and conceptual organization within those levels. Its contribution lies in shifting the analytical question from whether a level is procedural or conceptual to how procedural activity functions within the conceptual and epistemic structure characteristic of that level. The proposed trajectory, from perceptual-operational engagement through property-based analysis and relational coordination to deductive and axiomatic regulation, therefore provides a theoretical account of how procedural and conceptual knowledge may become progressively reorganized within geometric reasoning.

This interpretation, however, should remain theoretically cautious. The PCIT was derived from analysis of van Hiele descriptors rather than direct observation of learner development. It therefore cannot establish that all learners pass through the proposed forms of integration in a fixed or uniform sequence. Learners may display characteristics associated with different levels depending on the mathematical task, representation, topic, prior experience, or instructional context. The trajectory should therefore be viewed as a theoretically derived proposition requiring empirical examination rather than as a validated developmental scale. This limitation is particularly important because the present analysis did not involve an independent second coder, and the reliability of the proposed classification across independent judges remains to be established.

The findings also have implications for competency-oriented mathematics curricula. Uganda's Lower Secondary Mathematics Syllabus emphasizes mathematical concepts, skills, application, problem solving, and generic competencies (NCDC, 2019), while broader international curriculum developments similarly emphasize reasoning, problem solving, communication, and application alongside mathematical content (OECD, 2024). From the perspective of the PCIT, such competency-oriented approaches should not be interpreted as requiring the abandonment of procedural knowledge. Procedures remain important, but their instructional purpose should progressively change. Early procedural activity may support recognition and exploration, while later activity should increasingly support relational explanation, conjecturing, justification, proof, and reasoning within formal systems.

This has consequences for how progression is represented in curriculum design. Progression should not be understood only as increasing content difficulty or as mastery of increasingly complex procedures. It should also involve changes in what learners are expected to accomplish intellectually through those procedures. A construction activity, for example, may initially support recognition, later support investigation of properties, and eventually become a basis for conjecture or proof. Curriculum materials that repeatedly require learners to perform procedures without changing the reasoning demands attached to them may therefore provide limited support for progression toward higher levels of geometric thought.

The same principle applies to instructional design. Geometry tasks should not be classified simply as procedural or conceptual. More important is the relationship they establish between mathematical action and reasoning. Research combining exploratory learning with structured practice similarly indicates the value of designing learning experiences that support both conceptual and procedural knowledge development (Mavrikis et al., 2022). At Visualization, sorting, matching, tracing, folding, and manipulating figures can support recognition when learners are also required to describe what they notice. At Analysis, measurement, drawing, and construction can become tools for investigating properties. At Abstraction, these activities should increasingly support class relationships, conjectures, definitions, and informal justification. At Deduction, procedural steps should be governed by explicit warrants and proof structures, while at Rigor learners should reason about the consequences of assumptions within axiomatic systems. Thus, the instructional emphasis progressively shifts from successful execution of an action toward understanding, explaining, and justifying the mathematical relationships that regulate that action.

4.2. Conclusion

This study examined how procedural and conceptual knowledge are organised and coordinated across the five van Hiele levels of geometric reasoning. The analysis indicates that progression is better understood as a transformation in the function and organisation of procedural activity than as a simple movement from procedure to concept.

At Visualization, activity is predominantly perceptual and operational. At Analysis, procedures support the examination of properties. At Abstraction, procedural and conceptual activity become explicitly coordinated through representations, constructions, relationships, and informal justification. At Deduction and Rigor, procedural activity remains present but becomes increasingly embedded within formal conceptual and axiomatic structures.

The resulting Procedural–Conceptual Integration Trajectory (PCIT), developed by Ndungo and Akugizibwe, provides a more defensible and theoretically productive account of this relationship than a simple procedural-to-

conceptual continuum. Rather than positioning procedural and conceptual knowledge as competing alternatives, the PCIT suggests that geometry instruction should progressively reorganize procedural activity so that it increasingly supports explanation, relational reasoning, justification, proof, and abstraction.

4.3. Limitations of the Study

The study has important limitations. First, it analyses theoretical descriptors rather than learner responses, classroom interactions, or assessed performance. The PCIT therefore cannot be interpreted as evidence that individual learners necessarily display the proposed pattern. Second, the 25 descriptors are not independent, equally weighted measurements; they are textual characterizations of levels. Third, the present analysis does not provide an independent inter-rater reliability coefficient. Fourth, the interpretation focuses on the procedural-conceptual relationship and does not attempt to model all dimensions of mathematical proficiency, such as strategic competence, adaptive reasoning, or productive disposition.

5. Recommendations

5.1. For Classroom Practice

The PCIT suggests that geometry teaching should deliberately connect procedural activity with conceptual reasoning rather than treat them as separate forms of learning. Teachers should use activities such as measuring, constructing, drawing, sorting, and manipulating figures to support explanation, identification of relationships, conjecturing, justification, and proof. Lesson planning should therefore consider not only what learners do, but also the reasoning that the activity is intended to develop. Teacher education should prepare teachers to design tasks that progressively move learners from perceptual and property-based reasoning towards relational and deductive reasoning. Assessment should likewise capture both procedural performance and the conceptual reasoning underlying learners' solutions.

5.2. For Further Research

Future research should empirically test the PCIT using learner responses, classroom interactions, and geometry tasks designed to capture procedural-conceptual coordination. Studies should involve independent coders to strengthen analytical reliability and examine whether the proposed trajectory is observed across learners and contexts. Longitudinal research could investigate movement across the identified phases over time, while experimental or design-based studies could test whether instruction deliberately linking action, explanation, justification, and proof supports progression across van Hiele levels. Further studies may also examine the applicability of the PCIT across different geometry topics as well as other mathematics topics in different educational settings.

Ethics Statement: The study drew on published theoretical and curricular documents and did not involve new recruitment of human participants or analysis of identifiable personal data.

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